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THE EXPONENTIAL MAP FOR THE LAGRANGE PROBLEM ON DIFFERENTIABLE MANIFOLDS

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CONTENTS

A *parabolic space* $\mathscr{P} = \mathscr{P}(M, H, a)$ is defined as a C^∞ manifold M, a sub-bundle H of the tangent bundle T of M and a C^∞ symmetric, bilinear function $a: H \otimes H \to R$ which induces a positive-definite quadratic form on each fibre of H. A path $t \to f(t)$ in M is called *horizontal* if its tangent vector $\dot{f}(t)$ is everywhere in H. The Lagrange problem considered is that of finding, in the set $\Omega(P, Q)$ of piecewise C^1 horizontal paths in M which join fixed points P, Q, a path f_0 which minimizes the integral $\int a(\dot{f}(t) \otimes \dot{f}(t))\, dt$. Such an f_0 is called a *geodesic arc*. For each $x \in M$ there is an exponential map $e_x: T^*_x \to M$ of the set of covectors at x into M such that, for $y \in T^*_x$, $t \to e_x(ty)$ is geodesic, and also $e_x(N^*_x) = \{x\}$. Here, $N^* \subset T^*$ is defined by the exact sequence

$$0 \to N^* \to T^* \to H \to 0;$$

the epimorphism $T^* \to H$ being given by $y \to \tau_y$, where $y(\sigma) = a(\sigma \otimes \tau_y)$, $\sigma \in H$. The behaviour of e_x near N^*_x is studied and the following theorems are proved under the hypothesis (A) that, for every nonzero local section μ of N^* (a 1-form on M), $d\mu$ has maximal rank: (1) there is a neighbourhood U_x of the origin O_x of T^*_x such that $e_x | U_x \backslash N^*_x$ is diffeomorphic, (2) for every C^3 horizontal path $f: R \to M$ such that $f(0) = x$, there exists $\epsilon > 0$ such that $f | (-\epsilon, \epsilon)$ can be factorized in the form $e_x \tilde{f}_x$, where $\tilde{f}_x(0) = O_x$ and $\dot{\tilde{f}}_x(0)$ exists and is not tangential to N^*. The method of proof is to show (without hypothesis (A)) that $\mathscr{P}$ determines canonically a parabolic structure $\mathscr{P}'(M', H', a')$ on $M' = H_x \oplus N_x$ such that (primes being used for $\mathscr{P}'$ and x being identified with the zero of $H_x \oplus N_x$) e'_x is a first approximation to e_x near $N'^*_x \approx N^*_x$ when e_x, e'_x are compared in suitable charts. The geodesic properties of $\mathscr{P}'$ are readily computed and they lead to theorem (1) relative to $\mathscr{P}'$. The theorem $e_x \sim e'_x$ then allows this result to be carried over into $\mathscr{P}$. The approximate location of the set $e_x(U_x)$ is found in terms of a chart and it is proved that, for a path f as in (2), $f^{-1}e_x(U_x)$ is open. This, after further analysis, yields (2). In the course of the paper various related results are established. In particular, it is proved (3) without assumptions of normality that a sufficiently short geodesic arc is shorter than any other horizontal arc joining its end-points, (4) that, in a complete space $\mathscr{P}$, every pair of points P, Q for which $\Omega(P, Q)$ is not empty can be joined by a minimizing geodesic arc. Theorems (1) and (2) imply that a C^3 horizontal path f can be approximated by a geodesic polygon p_f which is homotopic to f by a standard homotopy of Morse theory. (No positive lower bound for the lengths of the sides of p_f is given—this would be a functional of the curvature of f.) As far as practicable, intrinsic notations are employed.

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Introduction

This paper was written in the hope that it might contribute to the application of Morse theory to the fixed end-point problem of Lagrange in the calculus of variations. For our purposes the latter may be described as follows. Let $\Omega = \Omega(P, Q)$ denote the set of piecewise C^1 paths f: $[0, 1] \to M$ joining P to Q in a C^∞ manifold M, where each f satisfies side conditions locally of the form

$$\mu^\lambda \circ \dot{f}(t) = 0 \quad (\lambda = m+1, \ldots, \dim M), \tag{1}$$

the μ^λ being 1-forms on M. Let T be the tangent bundle of M and F: $T \to R^+$ a function for which $F(\lambda y) \equiv |\lambda| F(y)$, $\lambda \in R$. The problem is to find an $f_0 \in \Omega$ for which $\int_0^1 F(\dot{f}(t))\,dt$ is minimized by $f = f_0$. Such an f_0 will be called a geodesic arc. The main results which we shall prove (§§ 7, 8) have, for their nearest analogue in Riemannian geometry, the theorem that each point a of a Riemannian manifold possesses a normal neighbourhood. Whereas, in Riemannian geometry, this result is an easy consequence of the non-singular character of the exponential map near the origin, here we have to deal with an exponential map which is singular—the type of degeneracy varying with the degree of non-integrability of the side conditions (1).† To simplify the problem, we suppose the function F to be (essentially) a Riemannian metric and the side conditions to be usually such that each μ^λ has its exterior derivative of maximal rank.

Recall that, in applying Morse theory to the geodesics of a Riemannian manifold M, one assumes without loss that the paths of Ω are parametrized proportionally to arc length and approximates to each $f \in \Omega$ by a geodesic polygon which is homotopic to f in a standard way (Milnor 1963). The existence of such a homotopy arises from the possibility of factorizing f locally through the exponential map, i.e. there exists an $\epsilon > 0$, independent of f, such that $f|(t-\epsilon l^{-1}, t+\epsilon l^{-1}) = e_{f(t)} \tilde{f}_t$, where l is the length of f and e_x is the exponential map at x. It will be proved (theorem 8·1) that such a factorization, in which ϵ now depends on f, can be carried out for the Lagrange problem. The reason for the dependence of ϵ on f here is that, if f were highly curved, its image could contain pairs of conjugate points arbitrarily close to each other, and an approximating geodesic polygon would have to possess a large number of sides in order to prevent some of them from being non-minimizing. The f-dependence of ϵ in the Lagrange problem might seem to rule out the possibility of applying Morse theory; however, in the author's view there are still grounds for optimism, since one can reason on the subset of Ω consisting of paths for which ϵ exceeds a fixed number.

As regards the formulation of the Lagrange problem, observe that, if F were a Riemannian metric, it would bring in redundant structure, because we are not interested in the lengths of paths which do not satisfy (1). It is not hard to see that a non-redundant structure is furnished by an epimorphism a: $T^* \to H$ of the cotangent bundle of M onto a sub-bundle H of the tangent bundle. The kernel, N^*, of a is then spanned locally by the μ^λ's while, for any $\tau \in H$, we have $F(\tau) = \sqrt{\tilde{\tau}(\tau)}$, where $\tilde{\tau} \in a^{-1}(\tau)$—the choice of $\tilde{\tau}$ being immaterial. Our object for study will therefore be the structure defined by an exact sequence

$$0 \to N^* \to T^* \xrightarrow{a} H \to 0, \quad H \subset T, \tag{2}$$

† If the Pfaffian system (1) is completely integrable, there is a foliation of M by Riemannian submanifolds of dimension m, and the set $\Omega(P, Q)$ may be empty. We are not concerned with this case.

together with certain hypotheses about rank, etc. This will be called a *parabolic structure* $\mathscr{P}(M, H, a)$ on M because the map a is a singular contravariant tensor a^{ij} for which the associated partial differential equation $a^{ij}\,\partial^2\phi/\partial x^i\,\partial x^j = 0$ is parabolic. M, with this structure, will be called a *parabolic space*. An example of a parabolic space is a fibre bundle M over a Riemannian space B, together with a connexion on M. The conditions (1) are satisfied by a path in M if it is horizontal, and the length of a horizontal path is that of its projection into B.

As already mentioned, the exponential map e_x: $T^*_x \to M$ (T^*_x = fibre over $x \in M$) for $\mathscr{P}(M, H, a)$ is singular, the 'kernel', $N^*_x \subset T^*_x$ mapping onto x under e_x. The central idea of this paper, which is developed in §§ 5, 6 and 7, is that $\mathscr{P}(M, H, a)$ determines canonically a parabolic structure $\mathscr{P}'$ on the vector space $H_x \oplus N_x$ such that (primes being used for $\mathscr{P}'$) N^*_x is the kernel for e'_x: $T'^*_x (= H^*_x \oplus N^*_x) \to H_x \oplus N_x$ and, when coordinates are properly chosen, e'_x gives a first approximation to e_x near N^*_x. The structure $\mathscr{P}'$ has geodesic properties which are readily computed (§§ 3, 4). Some of these properties, being true of a wider class of spaces, are proved in § 3. The main conclusion of § 4 is that there is a neighbourhood U'_x of the origin in T'^*_x such that $e'_x | U'_x \setminus N^*_x$ is diffeomorphic. The theorem $e_x \sim e'_x$, proved in § 6, then allows this result to be carried over unchanged into $\mathscr{P}$ (§ 7). The approximate location of the set $e_x(U_x)$ is found (§ 7) in terms of a chart and, in § 8, we show that, for every horizontal path f (i.e. satisfying (1)) through x, $f^{-1}e_x(U_x)$ is open—whence f factorizes locally in the form $e_x \tilde{f}_x$, where, if $x = f(t)$, $\tilde{f}(t)$ is the zero element of T^*_x. The tangent vector $q_f(t) \equiv \dot{\tilde{f}}_x(t)$ can be identified with a point of T^*_x, so the map $t \to q_f(t)$ gives a 'canonical lift' (theorem 8·2) of f into T^*_x. Crudely, this says that any C^3 horizontal path carries a set of Lagrange multipliers with which f satisfies the multiplier rule in case f is geodesic. These multipliers become large if f is highly curved. Finally, in § 8 a 'parabolic' version of covariant differentiation is given, for which scalar products are conserved under parallel translation. Geodesics are autoparallels, but not always vice versa.

Non-holonomic spaces, namely Riemannian or Finsler spaces carrying non-integrable Pfaffian systems, have been considered by Cartan (1953), Synge (1926), Rund (1954) and Vranceanu (1958) from a different point of view. Because of dynamical applications, Synge and Rund were mainly interested in what Synge calls *constrained geodesics*; these are the autoparallels of this paper (§ 8). A recent paper of Hermann (1962) is more closely related to this one, and we quote one of his results (proposition 2·7). Hermann's approach, however, is Lagrangian, whereas ours is Hamiltonian. They are locally equivalent, but ours seems more natural—the scene of most activity being T^* rather than $T \times R^{n-m}$.

1. Definitions and notation

Let X be a manifold; that is, connected, paracompact, C^∞, of finite dimension without boundary. An *m-dimensional Pfaffian system* ($0 < m < \dim X$) on X will be a sub-bundle HX (we do not distinguish notationally between a bundle and the bundle space) of the tangent bundle TX of X, where the fibre has dimension m. The cotangent bundle of X will be denoted by T^*X. An epimorphism α: $T^*X \to HX$ will be called *definite* if $y \circ \iota\alpha(y) \geqslant 0$ for all $y \in T^*X$ and further $y \circ \iota\alpha(y) = 0$ implies $y \in \ker \alpha$, where ι: $HX \subset TX$. We term α *symmetric* if $y_1 \circ \iota\alpha(y_2) = y_2 \circ \iota\alpha(y_1)$ whenever $\pi y_1 = \pi y_2$, π: $T^*X \to X$ being the projection.

DEFINITION 1·1. A *parabolic space*, $\mathscr{P}(M, H, a)$, is a manifold M, a Pfaffian system HM and a C^∞, definite, symmetric epimorphism a: $T^*M \to HM$.

NOTATION 1·1. For any $x \in X$, we write $T_x X$, $H_x X$, etc. for the fibre over x. We think of M as a fixed manifold and abbreviate TX, NX, ... to $T, H, \ldots$, respectively, whenever $X = M$. For example, TH_x will denote the tangent bundle of the vector space $H_x = H_x M$. The symbols π, ι (usually with diacritical marks) will always denote projection and inclusion maps respectively; their significance will not, therefore, be normally explained. In particular π unadorned will mean $T^* \to M$ throughout. The number 3·14...will be written Π. Inclusion maps will often be omitted. A vector $\tau \in TX$ will be termed *horizontal* (on X) if $\tau \in HX$. We write $N^* = \ker a$ and term $\mu \in T^*$ or $\lambda \in TT^*$ *null* if $\mu \in N^*$ or if $\lambda\ (\in TT^*_x)$ is the translate of a vector $\lambda' \in TN^*_x$, some $x \in M$. $\mathscr{F}^r_s X$ (or $\mathscr{F}^r_s$ if $X = M$) will denote the set of C^∞ sections in tensor bundles of type (r, s) over X, r and/or s being omitted when zero. The set of C^∞ sections $M \to N^*$ will be denoted by $\mathscr{N}$. If $V \subset M$, we call $\mu^{m+1}, \ldots, \mu^n \in \mathscr{N}$ a *basis for* $\mathscr{N}_V$ if $\mu^{m+1} \wedge \ldots \wedge \mu^n$ vanishes nowhere on V. If f: $X \to Y$ is C^1, f_*: $TX \to TY$ will denote the induced tangent map, and f^* its transpose. For any map f, $f(x)$ will sometimes be written $f|_x$; for example, if $\lambda \in \mathscr{F}_1$, $\tau \in \mathscr{F}^1$, we have $\lambda(\tau) \in \mathscr{F}$ and $\lambda(\tau)|_x$ denotes its value at $x \in M$. Also, $f|U$ will denote the restriction of f to U, and if F is a fibre bundle over X, $F|_x$ will denote the fibre over x. If V is a vector bundle and $\lambda \in R$, we write λ: $V \to V$, $v \to \lambda v$. If A is a vector space and $u, v \in A$, then v determines canonically a constant tangent field $\tilde{v}$, on A. We term v, $\tilde{v}|_u$ (and also v, $\tilde{v}$) *isomorphs* of each other. Except where indicated otherwise, Latin suffixes will range from 1 to $n (= \dim M)$ and suffixes $\alpha, \beta, \ldots, \epsilon$ from 1 to m, while $\kappa, \lambda, \ldots, \sigma$ will range from $m+1$ to n. The summation convention will be observed when a suffix occurs twice, once raised, once lowered. For a p-form α and a vector field λ we denote by $\mathrm{i}[\lambda]\alpha$ the $(p-1)$-form with components $(\mathrm{i}[\lambda]\alpha)_{j_2 \ldots j_p} = \lambda^{j_1}\alpha_{j_1 \ldots j_p}$. The Lie derivative of α with respect to λ is given by the Lichnerowicz formula

$$\mathscr{L}[\lambda]\alpha = \mathrm{i}[\lambda]\,\mathrm{d}\alpha + \mathrm{d}\mathrm{i}[\lambda]\alpha. \tag{1·1}$$

A circle or full stop will sometimes be used, as in $a \circ b$ or $r.s$ to indicate that a operates on b, or r multiplies $s (r, s \in R)$, if these marks are thought to be helpful. Any $\lambda \in \mathscr{F}^1$ defines a pseudogroup ϕ_t of transformations of M, and hence, by differentiation, pseudogroups ϕ_{t*} and ϕ^*_{-t} of T, T^*, respectively, for which $\pi'\phi_{t*} = \phi_t \pi'$, $\pi\phi^*_{-t} = \phi_t \pi$, π': $T \to M$. The vector fields on T, T^* which generate ϕ_{t*}, ϕ^*_{-t} will be called (following Hermann) the *first-order prolongations* of λ on T, T^*. The interval $[0, 1]$ will be denoted by I. Let $S \subset R$ and let f: $S \to X$ be C^0, piecewise C^1; we call f a *path* if S is a finite, closed interval, or a *curve* if S is a finite or infinite open interval. If $S = I$ we call f a *unit path*, and if f is a C^1 path whose tangent vector $\dot{f}(t)$ never vanishes we call f an *arc*. If $\dot{f}(t) \in H$ for all t, f will be described as *horizontal* and horizontal paths, arcs and curves will be called H-paths, H-arcs and H-curves, respectively.

If $y_1, y_2 \in T^*_x$, some $x \in M$, we call $a(y_1, y_2) = a(y_2, y_1) \overset{\text{def}}{=} y_1 \circ a(y_2)$ the *scalar product* of y_1 and y_2. Again, if $\tau_1, \tau_2 \in H_x$, $x \in M$, then $\tau_i = a(y_i)$, some $y_i \in T^*_x$, $i = 1, 2$, and we call

$$a(\tau_1, \tau_2) \overset{\text{def}}{=} a(y_1, y_2)$$

the scalar product of τ_1, τ_2. For $\tau \in T_x^*$ or $\tau \in H_x$, we write $|\tau| \overset{\text{def}}{=} \sqrt{a(\tau,\tau)}$. For any H-path f: $[b,c] \to M$, we define

$$J_q(f) = \int_b^c |\dot{f}(t)|^q \mathrm{d}t \quad (q=1,2) \tag{1·2}$$

calling $J_1(f), J_2(f)$ the *length* and *energy* of f, respectively. If U(open) $\subset M$, and if Ω_1, Ω_2 are the set of H-paths, unit H-paths, respectively, in U which join P, $Q \in U$, we say that f *minimizes* J_q *in* Ω_q if $f \in \Omega_q$ and $J_q(f) \leqslant J_q(h)$ for all $h \in \Omega_q$. In the cases where U, f have this property (1) for some U, (2) for $U = M$, we call f *relatively minimizing for* J_q, (*absolutely*) *minimizing for* J_q, respectively.

Proposition 1·1. For any $f \in \Omega_2$, $J_2(f) \geqslant \{J_1(f)\}^2$, equality holding if and only if f is parametrized proportionally to arc length. Consequently, if an arc f_1 minimizes J_1 in Ω_1, the unique $f_2 \in \Omega_2$ obtained by parametrizing f_1 proportionally to arc length, minimizes J_2 in Ω_2. Conversely, if f minimizes J_2 in Ω_2 it minimizes J_1 in Ω_1.

Proof. The first statement follows from Schwarz's inequality (Milnor, p. 70). To prove the last statement observe that f is necessarily parametrized by arc length, so that

$$J_2(f) = \{J_1(f)\}^2 = l^2, \quad \text{say.}$$

Suppose there exists $f' \in \Omega_1$ such that $J_1(f') = l' < l$. Without loss, assume $f' \in \Omega_2$. If $t \in I$ is the parameter for f', let $s(t)$ denote arc length for f' measured from $t = 0$, and define a new parameter τ by

$$(l'+\epsilon)\,\tau = s(t) + \epsilon t \quad (2\epsilon = l - l').$$

Let f' now be referred to this new parameter. We have

$$J_2(f') = \int_0^1 \left(\frac{\mathrm{d}s}{\mathrm{d}\tau}\right)^2 \mathrm{d}\tau = (l'+\epsilon)\int_0^1 \frac{\{s'(t)\}^2\,\mathrm{d}t}{s'(t)+\epsilon}$$

$$< (l'+\epsilon)\int_0^1 s'(t)\,\mathrm{d}t = l'(l'+\epsilon) < J_2(f),$$

contrary to hypothesis.

If f minimizes J_1 in Ω_1, f admits a representation as a C^1 arc with arc length as parameter (see the last paragraph of the proof of proposition 2·15). When this result is combined with proposition 1·1, one sees that the problems of minimizing J_q in Ω_q for $q = 1, 2$ are entirely equivalent. In practice, we shall generally apply standard calculus of variations techniques to J_2 rather than J_1.

Corollary 1·1. $\inf\limits_{f \in \Omega_2} J_2(f) = \{d_a(P,Q)\}^2$, where $d_a(P,Q) \overset{\text{def}}{=} \inf\limits_{f \in \Omega_2} J_1(f)$.

Definition 1·2. The Pfaffian system H is *locally horizontally connected* (l.h.c.) if, given a connected open set $U \subset M$ and $x, y \in U$, there exists an H-path in U joining x to y.

Proposition 1·2. If H is l.h.c. and μ $(\not\equiv 0) \in \mathscr{N}$, then $\mu \wedge \mathrm{d}\mu \not\equiv 0$.

Proof. If $\mu \wedge \mathrm{d}\mu \equiv 0$, $\mu \not\equiv 0$, there is, by Frobenius's theorem, an open $V \subset M$ and a $\phi \in \mathscr{F}V$ such that $\mathrm{d}\phi \not\equiv 0$, $\mu | V \not\equiv 0$ and $\mu \wedge \mathrm{d}\phi \equiv 0$. Hence, if $x, y \in V$ and $\phi(x) \neq \phi(y)$, no H-path in V exists which joins x to y.

Denote by $\mathscr{I}(N_x^*)$ the ideal generated by N_x^* in the graded algebra ΛT_x^* of exterior forms at x. Recall that m always denotes the dimension of the fibre of H.

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DEFINITION 1·3. The Pfaffian system H has *corank* h ($\leqslant m$ — write crk $H = h$) if h is the largest even integer such that, for all $x \in M$, the hypothesis $\mu \in \mathcal{N}_x$ and $(\mathrm{d}\mu)^{\frac{1}{2}h}|_x \in \mathcal{I}(N_x^*)$ implies $\mu|_x = 0$. In particular, crk $H = m$ if and only if for all $x \in M$,

$$\mu \in \mathcal{N}_x,\ \tau \in H_x,\ \mathrm{i}[\tau]\,\mathrm{d}\mu \in N_x^* \quad \text{implies} \quad \tau = 0 \quad \text{or} \quad \mu|_x = 0.$$

The condition crk $H = m$ is very restrictive and is analysed further in appendix 2. In particular, if crk $H = m < n-1$, we must have $m \equiv 0 \pmod 4$. The invariant crk H does not seem to have got into print before, though it must have been known to E. Cartan, for example. (A related invariant, given by the maximum r for which there exists $\mu \in \mathcal{N}$ such that $(\mathrm{d}\mu)^r \notin \mathcal{I}(N_x)$ was called by Cartan (1945, p. 107) the *Engel invariant* and by Vranceanu (1957, II, p. 219) the *rank* of the Pfaffian system.)

REMARK 1·1. Because of the resulting economy in notation, we shall often assume a Riemannian space under consideration to be complete. Local results (in particular, theorem 6·1) can, of course, be proved without this assumption.

2. GEODESICS

Let $U\{x^i\}$ be a coordinate neighbourhood of M, with (∂_i) the natural frame dual to $(\mathrm{d}x^i)$. For $y = y_i\,\mathrm{d}x^i$, $y' = y'_j\,\mathrm{d}x^j \in \mathcal{F}_1 U$, we have $a(y, y') = a^{ij} y_i y'_j$, where $a^{ij} = a(\mathrm{d}x^i, \mathrm{d}x^j)$. Again, if $q = q^i\,\partial_i$, $q' = q'^i\,\partial_i \in H_x$, $x \in U$, then $a(q, q') = a_{ij}\,q^i q'^j$, where $a_{ij} \in \mathcal{F}U$ is such that

$$a^{ir} a_{rs}\, a^{sj} = a^{ij}. \tag{2·1}$$

Hence, for any H-path $f\colon I \to U$, we have $\dot f(t) = \dot f^i(t)\,\partial_i$, and

$$J_2(f) = \int_0^1 a_{rs} f(t)\,.\,\dot f^r(t) \dot f^s(t)\,\mathrm{d}t. \tag{2·2}$$

If U is small enough there is a basis $(\mu^{m+1}, \ldots, \mu^n)$ for $\mathcal{N}_U$, so that $\dot f(t) \in H$ if and only if

$$\mu^\lambda(\dot f(t)) = \mu^\lambda_j f(t)\,.\,\dot f^j(t) = 0 \quad (\mu^\lambda = \mu^\lambda_j\,\mathrm{d}x^j). \tag{2·3}$$

These are the classical equations for the problem of minimizing J_2 in the set of unit H-paths in U joining fixed end-points. If f is relatively minimizing for J_2, then (Bliss 1963, pp. 207, 209) there exist 'canonical variables' $p_i\colon I \to R$ such that the functions $x^i f$, p_i satisfy the differential equations (multiplier rule)

$$\begin{aligned} \mathrm{d}x^i f(t)/\mathrm{d}t &= a^{ij} f(t)\,.\,p_j(t), \\ \mathrm{d}p_i(t)/\mathrm{d}t &= -\tfrac{1}{2} a^{rs}{}_{,i} f(t)\,.\,p_r(t)\,p_s(t), \end{aligned} \tag{2·4}$$

where $\mathrm{d}a^{ij} = a^{ij}{}_{,k}\,\mathrm{d}x^k$. Inspection of the Weierstrass–Erdmann corner condition (Bliss, p. 203) shows that f has no 'corners' and is thus a C^∞ H-arc (check that $|\dot f(t)|$ is constant and therefore nonzero if $f(0) \neq f(1)$). One verifies that the $p_i(t)$ are components of a covariant vector at $f(t)$.

DEFINITION 2·1. Let ω denote the *fundamental* 1-*form of* T^*, given, for $\tau \in T_y T^*$, by $\omega(\tau) = y(\pi_* \tau)$. Define $A \in \mathcal{F}\, T^*$ by $A(y) = |y|^2$. The *geodesic spray*, Θ ($\in \mathcal{F}^1 T^*$), is defined by

$$2\mathrm{i}[\Theta]\,\mathrm{d}\omega = -\mathrm{d}A. \tag{2·5}$$

By proposition 2·1 below, dω has maximal rank, so this definition is valid.

We have (Sternberg 1964, p. 199) for any $\alpha \in R$, $\alpha \neq 0$,

$$\alpha_* \Theta = \alpha^{-1}.\Theta, \quad \pi_*(\Theta|_y) = a(y). \tag{2·6}$$

We state without proof the following basic properties of ω.

Proposition 2·1. The 1-form ω on T^* is such that $(\mathrm{d}\omega)^n|_y \neq 0$ for all $y \in T^*$, so that there exists $\Omega \in \mathscr{F}^2 T^*$ for which $\Omega \circ \mathrm{d}\omega = 1_{TT^*}$. (In terms of Ω, we have $2\Theta = \Omega \circ \mathrm{d}A$.) For any $\sigma \in \mathscr{F}_1$, $\hat{\sigma} \overset{\text{def}}{=} \Omega(\pi^*\sigma)$ is a vertical vector field on T^*, constant on each fibre, and isomorphic to σ (notation 1·1). For any $\phi \in \mathscr{F}\, T^*$, $y \in T^*$ the following statements are true.

(i) $\hat{\sigma}(\phi)|_y = \mathrm{d}_t \phi(y + t\sigma\pi(y))|_{t=0} \quad (\mathrm{d}_t \equiv \mathrm{d}/\mathrm{d}t)$;

(ii) $\mathscr{L}[\hat{\sigma}]\,\omega = \pi^*\sigma$;

(iii) $\sigma^*\omega = \sigma$ (recall $\sigma: M \to T^*$);

(iv) if τ' is the first order prolongation on T* of a vector field τ on M, then $\mathscr{L}[\tau']\,\omega \equiv 0$;

(v) if M is a vector space over R, then $T^* \sim M \times M^*$. Let $\pi_1, \pi_2 : T^* \to M, M^*$, respectively, and let $\mathrm{t}_1: TM \to M$, $\mathrm{t}_2: TM^* \to M^*$ take tangent vectors to their isomorphs. Then, for any $u \in T^*$, $\sigma, \tau \in T_u\, T^*$,

$$\mathrm{d}\omega(\sigma, \tau) = (\mathrm{t}_2 \pi_2 \sigma) \circ (\mathrm{t}_1 \pi_1 \tau) - (\mathrm{t}_2 \pi_2 \tau) \circ (\mathrm{t}_1 \pi_1 \sigma).$$

Definition 2·2. Let $S \subset R$ and let $\lambda: S \to T^*$ be a Θ-orbit ($\dot{\lambda}(t) = \Theta|_{\lambda(t)}$ for all $t \in S$). If S is a finite or infinite open interval, $\pi\lambda$ will be called a *geodesic* (of $\mathscr{P}(M, H, a)$). If S is a finite, closed interval and $\dot{\lambda}(t) \neq 0$, $t \in S$, $\pi\lambda$ will be called a *geodesic arc*. If $\dot{\lambda}(t) \equiv 0$, we call $\pi\lambda$ (a constant map) a *null geodesic*.

Proposition 2·2 (*multiplier rule*). A necessary condition for the H-path $f: I \to M$ to be relatively minimizing for J_2 is that it be a geodesic arc.

Proof. Cover im$f$ by coordinate neighbourhoods $U\{x^i\}$. If $[b, c] \subset f^{-1}U$, then $f|[b, c]$ minimizes J_2 relatively in the set of H-paths $[b, c] \to U$ which join $f(b)$ to $f(c)$, and therefore satisfies (2·4). Now, the right-hand sides of (2·4) are just the components of Θ (given by (2·5)) in terms of the canonical chart $\pi^{-1}U\{\pi^* x^i, y_j\}$, where $y = y_j\, \mathrm{d}x^j$, evaluated at the point $p_i(t)\,.\mathrm{d}x^i|_{f(t)}$. Hence, the result.

Proposition 2·3. The spray Θ vanishes on N^*.

Proof. The non-negative function A (definition 2·1) vanishes on N^*, whence $\mathrm{d}A|N^* = 0$. By (2·5) and because $(\mathrm{d}\omega)^n \neq 0$, we have $\Theta|N^* = 0$.

Corollary 2·1. Any constant map $R \to M$ is a (null) geodesic.

Definition 2·3. A *compatible Riemannian metric* (c.R. metric) on M is a positive-definite Riemannian metric g: T* $\to T$ such that (with notation analogous to that for a) $x \in M$, σ, $\tau \in H_x$ implies $g(\sigma, \tau) = a(\sigma, \tau)$. ($g^{-1}|H$ is a splitting of the exact sequence (2).)

Definition 2·4. Let f be a geodesic; if f is also geodesic for a c.R. metric g on M, g will be said to *admit* f. (A geodesic of g will be the projection of an orbit of (2·5) where A has been replaced by the function $y \to g(y, y)$.)

Proposition 2·4. There exists a c.R. metric on M.

Proof. Let $\tilde{g}$ be a Riemannian metric on M; $\tilde{g}$ determines, by orthogonality, a decomposition $T = H \oplus H^{\perp}$, with projections π_1, π_2. Let $\mathrm{g}(\xi, \eta) \equiv a(\pi_1 \xi, \pi_1 \eta) + \tilde{g}(\pi_2 \xi, \pi_2 \eta)$.

A c.R. metric is thus the sum of parabolic 'metrics' associated with supplementary Pfaffian systems H, $H^{\perp}$.

Proposition 2·5. Let S (closed) $\subset M$ and let g_0 be a c.R. metric on U (open), where $S \subset U \subset M$. Then there exists a c.R. metric g on M such that $g|S = g_0$.

Proof. Use a partition of unity argument, noting that, if g_i $(i = 1, \ldots, N)$ are c.R. metrics, so is $\Sigma\phi_i g_i$, where $\{\phi_\alpha\}$ is a p.u.

Proposition 2·6. Let f be a non-null geodesic arc without self-intersections. There exists a c.R. metric on M which admits f.

Proof. We have $f = \pi\gamma$, where $\gamma: I \to T^*$ is an embedding. Let $\tilde{g}$ be a c.R. metric on M, so that $a \circ \tilde{g}^{-1}|H = 1_H$. There exists U (open) $\subset M$ (e.g. a tubular neighbourhood of im f) and a C^∞ section $\psi: U \to T^*$ for which $\psi f = \gamma$. Hence, using a suffix U to denote $|U$, we have a decomposition $T^*_U = N_1 \oplus N_2$, where $N_1 = N^*_U$, $N_2 = \Psi \oplus L$, Ψ, L being defined as follows. Ψ (with fibre R) is the sub-bundle of T^*_U which contains im ψ, and L is the orthogonal complement (relative to $\tilde{g}$) of $N^*_U \oplus \Psi$ in T^*_U. Let $T_U = H_1 \oplus H_2$, where $H_2 = H_U$, be the decomposition dual to $N_1 \oplus N_2$, and write $\pi_i: T^*_U \to N_i$, $\pi'_i: T_U \to H_i$, $i = 1, 2$. Set $g = \pi'_1 \tilde{g}\pi_1 + a$: $T^*_U \to T_U$. Then $g(\nu) = 0$ implies $a(\nu) = \pi'_1\tilde{g}\pi_1(\nu) = 0$, whence $\nu \in N^*_U$, $\tilde{g}\pi_1(\nu) \in H_U$, so that $\nu = 0$—$\tilde{g}(N^*)$ being supplementary to H. Thus, g is an isomorphism. Next, for any $\sigma \in T^*_U$,

$$\sigma \circ g(\sigma) = \sigma \circ \pi'_1 \tilde{g}\pi_1(\sigma) + \sigma \circ a(\sigma) = (\pi_1 \sigma) \circ \tilde{g}(\pi_1 \sigma) + \sigma \circ a(\sigma), \tag{2·7}$$

which is non-negative, whence g is positive-definite. Now, $g|N_2 = a|N_2$: $N_2 \approx H_2$, so $ag^{-1}|H_U = 1$, and g is a c.R. metric on U. Again, by (2·7) and definition 2·1

$$G(\sigma) - A(\sigma) = \sigma \circ (g(\sigma) - a(\sigma)) = \tilde{g}(\pi_1 \sigma, \pi_1 \sigma) \geqslant 0,$$

where $G(\sigma) \overset{\text{def}}{=} g(\sigma, \sigma)$. Hence, since $\pi_1(\Psi) = 0$, the non-negative function $G - A \in \mathscr{F}\, T^*_U$ vanishes on Ψ and we have $(\mathrm{d}G - \mathrm{d}A)|\Psi = 0$. Accordingly, by (2·5), $2\mathrm{i}[\Theta]\,\mathrm{d}\omega = -\mathrm{d}G$ at all points $y \in \mathrm{im}\,\gamma$. Thus, f is geodesic relative to g and the result follows from proposition 2·5. From a known result of Riemannian geometry, we have

Corollary 2·2. If f: $(a, b) \to M$ is a geodesic of $\mathscr{P}(M, H, a)$, each $t \in (a, b)$ has a neighbourhood $[t - \epsilon, t + \epsilon]$ for which the unit H-path $s \to f(t - \epsilon + 2\epsilon s)$ has smaller energy than any other unit H-path joining its end-points.

Definition 2·5. Let g be a c.R. metric on M. For any $x, y \in M$, $d_a(x, y)$ will be as in corollary 1·1 and $d_g(x, y)$ will denote the infimum of the lengths (relative to g) of all paths in M which join x to y.

Provided that all pairs $x, y \in M$ can be joined by an H-path, d_a defines a topological metric for M, and we have $d_g(x, y) \leqslant d_a(x, y)$ for all x, y. Hence, a Cauchy sequence for d_a is also one for d_g, so the following proposition follows from a result of Hermann (1962, § 4).

Proposition 2·7. Let $\mathscr{P}(M, H, a)$ be complete; viz. all Cauchy sequences relative to d_a converge. Let f: $(b, c) \to M$, $b, c \in R$, be a geodesic. Then $f = h|(b, c)$, where h: $R \to M$ is a geodesic.

Our next objective is proposition 2·13. We shall require some background material, of which a full treatment is given by Bliss (1963, part II). The reader will be able to translate the results given there into the present context by means of

PROPOSITION 2·8. Let f be a geodesic arc. There is a coordinate neighbourhood $U\{x^i\} \subset M$ containing im f and a C^∞ map $\mu: U \to GL(n)$ with the property that $(\mu(x))^\nu_j \, dx_j$ is a basis for $\mathcal{N}_U$.

Proof. A proof of the first assertion is given by Morse (1934, p. 109). The second assertion is proved likewise by piecing together maps μ defined locally, after premultiplication by suitable constant maps into $GL(n)$.

DEFINITION 2·6. The geodesic arc f: $[a, b] \to M$ is *normal* if there is exactly one Θ-orbit ρ: $[a, b] \to T^*$ such that $f = \pi\rho$. (This agrees with the classical definition (Bliss 1963, § 77).)

DEFINITION 2·7. We call $\xi \in H_x$ *variationally normal* if $\mu \in \mathcal{N}_x$ and $\mathrm{i}[\xi] \, d\mu \in N_x$ imply $\mu|_x = 0$. H will be called variationally normal if, for all $x \in M$, there is a variationally normal $\xi \in H_x$.

PROPOSITION 2·9. If $\xi \in H_x$ is variationally normal, every geodesic arc f: $[b, c] \to M$ such that $\dot{f}(c) = \xi$ is normal.

Proof. Since a geodesic arc is normal if it contains a normal subarc, we may assume f arbitrarily short. Let f_1, f_2: $[b, c] \to T^*$ be distinct Θ-orbits such that $\pi f_1 = \pi f_2 = f$. From (2·6), we have

$$\dot{f}(t) = \pi_* \dot{f}_1(t) = af_1(t) = af_2(t),$$

whence $(f_1 - f_2)$: $[b, c] \to N^*$. We embed $f_1 - f_2$ in a C^∞ section μ: $U \to N^*$, where U(open) $\ni$ im f, so that $f_1 = f_2 + \mu f$. Let λ: $\pi^{-1}U \to \pi^{-1}U$ be the diffeomorphism $y \to y + \mu\pi y$, so that $\lambda^*\omega = \omega + \tilde{\mu}$, $\lambda^* A = A$, where $\tilde{\mu} = \pi^*\mu$. For any $\sigma \in T_{f_2(c)} T^*$,

$$\begin{aligned} dA(\sigma) &= dA(\lambda_* \sigma) = 2\, d\omega(\lambda_* \sigma, \dot{f}_1) = 2\, d\omega(\lambda_* \sigma, \lambda_* \dot{f}_2) \\ &= 2\, d(\lambda^*\omega)\, (\sigma, \dot{f}_2) = 2(d\omega + d\tilde{\mu})\, (\sigma, \dot{f}_2) = dA(\sigma) + 2\, d\tilde{\mu}(\sigma, \dot{f}_2), \end{aligned}$$

where $\dot{f}_1, \dot{f}_2$ mean $\dot{f}_1(c), \dot{f}_2(c)$. Since σ is arbitrary, $\mathrm{i}[\dot{f}_2] \, d\tilde{\mu} = 0$, whence, by definition 2·7, $\mu|_x = f_1(c) - f_2(c) = 0$. Since no two Θ-orbits intersect, the proposition follows.

PROPOSITION 2·10. If f is a geodesic arc and $\mu \in \mathcal{N}_{\mathrm{im} f}$ is such that $\mu | \mathrm{im} f \not\equiv 0$ and $\mathrm{i}[\dot{f}(t)] \, d\mu = 0$ for all t, then there is a 1-parameter family of Θ-orbits in T^* which all have f for their projection under π.

Proof. Let $f = \pi f_1$, where f_1 is a Θ-orbit. The section μ: $U \to T^*$, $U = \mu^{-1} T^*$, determines a 1-parameter group of diffeomorphisms

$$\Lambda: \pi^{-1}U \times R \to \pi^{-1}U, \quad (y, r) \to y + r\mu\pi y.$$

By a similar calculation to the preceding one, we find that

$$\Lambda_{r*}(\Theta|_y) = \Theta|_{\Lambda_r y}, \quad \Lambda_r = \Lambda|\pi^{-1}U \times \{r\},$$

whenever y is a point at which $\mathrm{i}[\Theta] \, d(\pi^*\mu) = 0$. All points $f_1(t)$ being of this type, it follows that $\Lambda_r f$ is a Θ-orbit for each r.

We assume henceforth that M is complete (cf. remark 1·1).

DEFINITION 2·8. Through each point $u \in T^*$ construct the Θ-orbit f_u: $R \to T^*$ for which $f_u(0) = u$. Define E: $T^* \to T^*$, $u \to f_u(1)$, $e = \pi E$, $E_o = E | T^*_o$, $e_o = \pi E_o$, where $o \in M$. We call e_o the *exponential map at o*. A point of T^*_o at which e_o is singular will be called *conjugate*.

Clearly, E, E_o, e, e_o are C^∞. For a geodesic arc f: $[b, c] \to M$ one often calls $f(c)$ conjugate to $f(b)$, meaning $f = e_{f(b)}\xi$ (cf. corollary 2·3), where $\xi(t) = (t-b)\,(c-b)^{-1}\xi(c)$ and $\xi(c)$ is conjugate.

PROPOSITION 2·11. Let $f_1: R \to T^*$ be a Θ-orbit. Then $f_\lambda: R \to T^*$, $t \to \lambda f_1(\lambda t)$, $\lambda \in R$, is a Θ-orbit.

Proof. We have $f_\lambda = \beta f_1 \alpha$, where

$$R \xrightarrow{\alpha} R \xrightarrow{f_1} T^* \xrightarrow{\beta} T^*, \quad t \xrightarrow{\alpha} \lambda t, \quad y \xrightarrow{\beta} \lambda y.$$

Let d_t denote $\mathrm{d}/\mathrm{d}t$ and let $\phi: R \to R$ be C^1; then $(\alpha_* \mathrm{d}_t)\phi = \mathrm{d}_t \phi(\lambda t) = \lambda \mathrm{d}_\tau \phi(\tau)$, $\tau = \lambda t$. Hence, by (2·6),

$$\dot{f}_\lambda(t) = \beta_* f_{1*} \alpha_* \mathrm{d}_t = \lambda \beta_* f_{1*} \mathrm{d}_\tau = \lambda \beta_* (\Theta|_{f_1(\tau)}) = \Theta|_{f_\lambda(t)}.$$

COROLLARY 2·3. Let $f: R \to T^*$ be a Θ-orbit such that $f(0) \in T^*_o$. Then, for all $t \in R$,

$$tf(t) = E_o\{tf(0)\}, \quad \pi f(t) = e_o\{tf(0)\}.$$

Let a 1-*parameter family of H-paths* in M be a map $h: I_1 \times R \to M$, $I_1 = [\alpha, \beta]$, $\alpha, \beta \in R$, such that (*a*) $u \to h(u,v)$ is an H-path for each v, (*b*) for each v, the path in T, $u \to h_* \circ \partial/\partial v$, is piecewise C^1 and for all $\phi \in \mathscr{F}$, the derivatives $\partial^2(\phi h)/\partial v^2$ are C^0 in u. There is a natural diffeomorphism of period 2, ψ: $TT \to TT$, which arises from the commutativity of partial differentiation in M. Let a *variation* of a geodesic arc $\gamma: I_1 \to M$ be a path $\eta: I_1 \to T$ such that $\eta(\alpha) = \eta(\beta) = 0$, $\pi'\eta = \gamma$ $(\pi': T \to M)$, and $\psi\dot{\eta}$ is everywhere tangential to H. In particular, if h (above) is such that $h(u,0) = \gamma(u)$, each $u \in I_1$, and if $h(\{u\} \times R) = \gamma(u)$ when $u = \alpha$ or β, then $u \to h_*(\partial/\partial v|_{(u,0)})$ is a variation of γ. The crucial property of normal geodesic arcs is that every variation can be realized in this way. If η is a variation of γ and f is a 1-parameter family which realizes η, we have a function $\phi: R \to R$, $v \to J_2(f|I_1 \times \{v\})$ for which the 'first variation', $\dot{\phi}(0)$, is zero. Also, the 'second variation', $\ddot{\phi}(0)$, is independent of the choice of realization f of η, and is non-negative if γ is relatively minimizing. If $\ddot{\phi}(0) = 0$ for some η_0 in the set of non-zero variations of γ, η_0 must be an extremal of the so-called 'accessory problem' (Bliss, § 81). The multiplier rule, together with the Weierstrass–Erdmann corner condition, then show that η_0 must be a Jacobi field over γ, namely, in this case, a C^1 path of the form $\tilde{\eta}: I_1 \to T$, where $\tilde{\eta}(\alpha) = \tilde{\eta}(\beta) = 0$, $\tilde{\eta}(u) = \tilde{f}_*(\partial/\partial v|_{(u,0)})$, $\tilde{f}$ being a 1-parameter family of geodesic arcs such that $\tilde{f}(\{\alpha\} \times R) = \gamma(\alpha)$. Conversely, the second variation vanishes for such a Jacobi field. Call a Jacobi field $t \to \eta(t)$ *trivial* if $\eta(t) = 0$ for all t.

LEMMA 2·1. If there is a C^2 map $k: \mathscr{I} \times R \to T^*$, $\mathscr{I} = [b, c]$, such that

$$\begin{aligned} &(1) \quad k_*(U) = \Theta|_{k(u,v)} \neq 0 \quad \text{for all } u, v, \\ &(2) \quad \pi_* k_*(V) = 0 \quad \text{for all } u \in \mathscr{I},\ v = 0, \\ &(3) \quad \pi_* k_*(U|_{(b,0)}) \text{ is variationally normal,} \end{aligned}$$

where $U = \partial/\partial u$, $V = \partial/\partial v$, then $k_*(V) = 0$ for all $u \in \mathscr{I}$, $v = 0$.

Proof. By (2), $k_*(V)$ is vertical when $v = 0$; let $\hat{V} = \hat{V}(u)$ denote its isomorph. Choose $\kappa \in \mathscr{F}$; by (2·6), we have

$$VU(k^*\pi^*\kappa)|_{v=0} = V(k^*\Theta(\pi^*\kappa)) = Va(k(u,v), \quad \mathrm{d}\kappa|_{\pi k(u,0)}) = a(V(u), \mathrm{d}\kappa),$$

the function on the left being zero, since $[U, V] \equiv 0$ and $\pi_* k_* V|_{v=0} = 0$. Hence, κ being arbitrary, $\hat{V}$ is null. Now choose $\mu \in \mathscr{N}$ such that $\mu|_{\pi k(u,0)} = \hat{V}(u)$, each u, and define (cf. proposition 2·1) $\hat{\mu} = \Omega(\pi^*\mu)$. Since U, Θ and V, $\hat{\mu}$ are k-related,

$$[\hat{\mu}, \Theta] = k_*[U, V] = 0 \quad \text{and} \quad \mathscr{L}[\hat{\mu}] A = \hat{\mu} A = 0.$$

Hence, by (2·5) and proposition 2·1, we have at $k(b, 0)$,

$$O = \mathscr{L}[\hat{\mu}]\,\mathrm{i}[\Theta]\,\mathrm{d}\omega = \mathrm{i}[[\hat{\mu}, \Theta]]\,\mathrm{d}\omega + \mathrm{i}[\Theta]\,\mathrm{d}(\pi^*\mu) = \pi^*\{\mathrm{i}[\pi_* k_*(U)]\,\mathrm{d}\mu\},$$

whence, by hypothesis, $\mu|_{\pi k(b,0)} = 0$. Thus, $k_*(V)$ vanishes at $k(b, 0)$. Recall that the components of $k_*(V)$ satisfy first order, linear, homogeneous differential equations in u, so that the vanishing of $k_*(V)$ at $u = b$ implies its vanishing for all u.

PROPOSITION 2·12. If $y \in T_o^*$ is conjugate, and if $a(y)$ is variationally normal, there is a non-trivial Jacobi field over the geodesic arc $I \to M$ $t \to e_o(ty)$.

Proof. Since y is conjugate, there exists $z \in T_y T_o^*$ such that $e_{o*}(z) = 0$, $z \neq 0$. Let $\hat{z}$ be the isomorph of z and define the 1-parameter family of geodesics

$$f\colon I \times R \to M,\ (u, v) \to e_o\{u(y + v\hat{z})\}.$$

Then $u \to f_*(V|_{(u,0)})$ is a Jacobi field. Assume it is trivial and let $k\colon I \times R \to T^*$ be a map as in lemma 2·1, where $u \to k(u, v)$ is the Θ-orbit through $y + v\hat{z}$. By corollary 2·3, for $u \neq 0$, we have $k(u, v) = u^{-1}E_o\{u(y + v\hat{z}\}$. Then $\pi k = f$ and, by (2·6), $\pi_* k_*(U|_{(0,0)}) = a(y)$ is variationally normal. Conditions (1), (2), (3) of lemma 2·1 are thus satisfied, whence

$$k_*(V|_{(1,0)}) = E_{o*}(z) = 0.$$

Since E_o is a diffeomorphism, this gives $z = 0$, contrary to hypothesis.

PROPOSITION 2·13. Let $\gamma\colon I_1 \to M$, $I_1 = [\alpha, \beta]$ be a geodesic arc such that $\gamma(\lambda)$ is conjugate to $\gamma(\alpha)$, where $\lambda \in (\alpha, \beta)$. If $\dot{\gamma}(\beta)$ is variationally normal, γ is not relatively minimizing.

Proof. Suppose to the contrary and that, by proposition 2·12, there exists a non-trivial Jacobi field $\eta_2\colon I_2 \to T$, $I_2 = [\alpha, \lambda]$ over $\gamma|I_2$. Then the map $\eta_1\colon I_1 \to T$, given by

$$\eta_1|I_2 = \eta_2, \quad \eta_1(u) = 0|_{\gamma(u)}, \quad u \in [\lambda, \beta],$$

is a variation whose second variation vanishes. It is therefore a Jacobi field. Let $k\colon I_1 \times R \to T^*$ be a one-parameter family of Θ-orbits which realizes η_1; then $k|[\lambda, \beta]$ is a map as in lemma 2·1, whence $k_*(V) = 0$ when $u \in [\lambda, \beta]$, $v = 0$. By the last remark in the proof of lemma 2·1, $k_*(V)$ therefore vanishes also on $I_1 \times \{0\}$.

PROPOSITION 2·14. If M is variationally normal, M is l.h.c. Also, $d_a(x_1, x_2) \to 0$ as $x_2 \to x_1$, the limit being uniform on any compact subset of M.

Proof. Let U(open) $\subset M$ and let $f\colon I \to U$ be continuous. We have to construct an H-path joining the end-points b, c of f. At each $z \in \operatorname{im} f$, there is a $\xi \in H_z$ and a geodesic arc g_z issuing from z in the direction ξ. By corollary 2·2, if g_z is sufficiently short, it is minimizing and is contained in U, so, by proposition 2·13, its end-points ζ, z are non-conjugate. By definition 2·8, there is a neighbourhood $Q_z \subset U$ of z such that all points of Q_z can be joined to z by a geodesic arc in U. By compactness there is a finite set $\zeta_1, z_1, \ldots, \zeta_r, z_r$, where $z_1 = b$, $z_r = c$, $z_1, \ldots, z_r \in \operatorname{im} f$, such that the corresponding Q's cover $\operatorname{im} f$. Hence, the broken geodesic arc $z_1 \zeta_1 z_2 \zeta_2 \ldots z_r$ is in U and joins b to c.

Next, let $U \subset M$ be compact. Given $\epsilon > 0$, we construct as above for each point $z \in U$ a point ζ such that the geodesic arc ζz has length $< \epsilon$. Then z has a neighbourhood Q_z for which each pair $z_1, z_2 \in Q_z$ can be joined by a broken geodesic of length $< 3\epsilon$. Let g be a c.R. metric for M and let δ be the Lebesgue number of the cover $\{Q_z\}_{z \in U}$ relative to the metric d_g. Then we have $d_a(x_1, x_2) < 3\epsilon$ whenever $d_g(x_1, x_2) < \delta$, $x_1, x_2 \in U$.

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COROLLARY 2·4. If M is variationally normal, the topology of M induced by d_a is that of the underlying manifold.

PROPOSITION 2·15. If $\mathscr{P}(M, H, a)$ is complete and l.h.c., given any $z_1, z_2 \in M$, there is a minimizing geodesic arc which joins z_1 to z_2.

Proof. Denote by Ω the set of unit H-paths in M which join z_1 to z_2 and by $\Omega'(\supset \Omega)$ the set of absolutely continuous maps $I \to M$ which join z_1 to z_2 and whose tangent vectors are horizontal p.p. Set

$$l' = \inf_{f \in \Omega'} J_1(f) \leqslant l = d_a(z_1, z_2).$$

There exists $f_o \in \Omega'$ for which $J_1(f_o) = l'$. For, suppose to the contrary; then there is a sequence $\{f_i\}$ in Ω' such that $J_1(f_i) \to l'$. By dividing I into 2^N $(N = 1, 2, \ldots)$ subintervals and applying a classical argument (Cartan 1951, note IV) using proposition 2·14, we arrive at a continuous map $\phi: I \to M$ with the properties that

$$\phi(0) = z_1, \quad \phi(1) = z_2, \inf_{f \in \Omega'_t} J_1(f) = l'\,|t'-t|,$$

where Ω'_t is the set of a.c. maps $[t, t'] \to M$ which are horizontal p.p. and which join $\phi(t)$ to $\phi(t')$. Let g be a c.R. metric and, for each $t \in I$ choose a solid open sphere $\mathscr{S}_{2\delta}$ of radius 2δ relative to g and centre $\phi(t)$ such that there is a chart $\chi: U\{x^i\} \to R^n$ for which $U \supset \bar{\mathscr{S}}_{2\delta}$. Choose $t' \in (t, t+\delta/l)$ and let Ω'_t be as above, but restricted to maps f for which

$$J_1(f) \leqslant 2l'(t'-t).$$

By construction, Ω'_t is not empty, and its elements have their images in $\mathscr{S}_{2\delta}$. By means of the chart χ we convert the problem of minimizing J_1 in Ω'_t to a variational problem in R^n with equations analogous to (2·2) and (2·3). The set $\chi\bar{\mathscr{S}}_{2\delta}$ being compact and the 'paths' $\chi\Omega'_t$ being of bounded length, there is a generalized curve† (McShane 1940*a*, theorem 6·3)

$$C_o: \{y_o(\tau), \quad \mathscr{M}[\tau, \Phi]\},$$

which joins $\chi\phi(t)$ to $\chi\phi(t')$, which satisfies (2·3) for almost all τ and for all vectors r carried, and for which the integral

$$J_1(C_o) \equiv \int_t^{t'} \mathscr{M}[\tau, \surd(a_{ij}\, r^i r^j)]\, \mathrm{d}\tau$$

is a minimum. Because the equations $\mu_i^\lambda r^i = 0$ are linear and because the function

$$r \to \surd(a_{ij}\, r^i r^j)$$

is convex on the set $\{r \in R^n | \mu_i^\lambda r^i = 0,\ \lambda = m+1, \ldots, n\}$, C_o is the isomorph of an ordinary curve (McShane 1940*a*, theorem 11·1). Accordingly,

$$J_1(C_o) = \int_t^{t'} \surd(a_{ij}\, \dot{y}_o^i(\tau)\, \dot{y}_o^j(\tau))\, \mathrm{d}\tau = l'(t'-t).$$

Going back to M and piecing together a finite number of maps of the form $\chi^{-1}y_o$, we conclude the existence of an element $f_o \in \Omega'$ of length l'.

† The reader is referred to McShane's papers for the definitions and concepts introduced here.

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In each coordinate neighbourhood U, f_o must satisfy the multiplier rule (McShane 1940*b*, theorem 10·1)

$$F_{r^i}(y_o(\tau), \dot{y}_o(\tau), \lambda(\tau)) = c_i + \int_t^\tau F_{y^i}(y_o(\tau'), \dot{y}_o(\tau'), \lambda(\tau'))\,\mathrm{d}\tau' \tag{2·8}$$

for almost all τ, for some constants c_i and for some continuous functions $\lambda_{m+1}, \ldots, \lambda_n$. Here $F(y, r, \lambda)$ is the function

$$\sqrt{(a_{ij}(y)\, r^i r^j)} + \lambda_\nu \mu_i^\nu(y)\, r^i$$

and suffices denote partial differentiation. Also, τ can be taken to be a 'standard parameter' (McShane 1940*a*, lemma 7·1), so that $\Sigma\{\dot{y}_o^i(\tau)\}^2 = 1$, p.p. The equations

$$F_{r^i}(y, r, \lambda) = p_i, \quad \mu_i^\lambda r^i = 0 \quad (\Sigma(r^i)^2 = 1)$$

yield r, λ as C^∞ functions of y, p. Hence, taking for p_i the right-hand side of (2·8), we obtain r, λ as C^1 functions of τ. Accordingly, by (2·8), $y_o^i(\tau)$ is C^2 and f_o is geodesic.

3. Parabolic structures associated with Riemannian spaces

Let B be a Riemannian space ($\dim B = m$) with metric b, let M be a principal fibre bundle over B and Γ a connexion on M. There is a parabolic structure $\mathscr{P}(M, H, a)$ on M, where $a(\sigma, \tau) = b(\pi_{1*}\sigma, \pi_{1*}\tau)$, $\pi_1 \colon M \to B$, and σ, τ ($\in H_x M$, some $x \in M$) are horizontal in the usual sense for Γ. Let $x, y \in B$ and let $\Omega(x, y)$ be the set of unit paths in B which join x to y. There is an equivalence relation $\mathscr{R}(x, y)$ on $\Omega(x, y)$ given by: $f_1 \sim f_2$ if there exist H-paths $\tilde{f}_1$, $\tilde{f}_2$ in M over f_1, f_2, respectively, such that $\tilde{f}_1(0) = \tilde{f}_2(0)$, $\tilde{f}_1(1) = \tilde{f}_2(1)$. In particular, $\Omega(x, x)/\mathscr{R}(x, x)$ is the holonomy group of Γ minus the group structure. The variational problem of finding in each element $\xi \in \Omega(x, y)/\mathscr{R}(x, y)$ a representative path of shortest length is of Lagrange type and is isomorphic to that of finding a minimizing geodesic of $\mathscr{P}(M, H, a)$ joining two suitable fixed points of M.

We study the following situation. Σ_o is a vector space over R and Σ denotes affine space, where $\dim \Sigma = \dim \Sigma_o < m$ and Σ_o acts on Σ by translation. Thus, $M \equiv B \times \Sigma$ is a trivial principal fibre bundle over B with fibre Σ and (Abelian) group Σ_o. Let $\delta \colon \Sigma_o^* \to \mathscr{F}_1 B$ be a monomorphism. If X_1, X_2, are manifolds and $\tau \in T_{(x_1, x_2)} X_1 \times X_2$, $\rho \in T^*_{(x_1, x_2)} X_1 \times X_2$, we write $\tau = (\tau_1, \tau_2)$, $\rho = (\rho_1, \rho_2)$, where $\tau_\alpha \in T_{x_\alpha} X_\alpha$, $\rho_\alpha \in T^*_{x_\alpha} X_\alpha$, $\alpha = 1, 2$; also, we say that τ is *tangential to* X_α if $\tau_{3-\alpha} = 0$. Let $\pi_1, \pi_2 \colon B \times \Sigma \to B, \Sigma$, respectively, and let $\mathrm{t} \colon T\Sigma \to \Sigma_o$ map $\tau \in T\Sigma$ to the element of Σ_o which generates it. For any $\tau \in TB$ or $\tau \in T^*B$, we define $\|\tau\| = \sqrt{b(\tau, \tau)}$.

Proposition 3·2. There is a connexion Γ on $M \equiv B \times \Sigma$ determined by a 1-form γ on M with values in Σ_o, defined as follows. For any $\sigma \in \Sigma_o^*$, $(\tau_1, \tau_2) \in TM$,

$$\sigma \circ \gamma(\tau_1, \tau_2) = \sigma \mathrm{t}(\tau_2) - \delta\sigma(\tau_1). \tag{3·1}$$

Proof. γ is C^∞, and τ tangential to Σ implies $\gamma(\tau) = \mathrm{t}(\tau)$, while, if $\rho_\sigma \colon M \to M$ denotes (right) translation by $\sigma \in \Sigma_o$, $\gamma(\rho_{\sigma*}\tau) = \gamma(\tau) = (\operatorname{ad}\sigma) \circ \gamma(\tau)$, so γ is a connexion (cf. Lichnerowicz 1955, p. 58).

Notations 3·1. $\mathscr{P}\{B, \Sigma_o, \delta, b\}$ will denote the parabolic structure $\mathscr{P}(M, H, a)$ where $M = B \times \Sigma$ (Σ as above), $H = HM$ is the set of horizontal vectors relative to

$$\Gamma(\tau \in H \text{ iff } \gamma(\tau) = 0),$$

and if $x \in M$, $\tau, \tau' \in H_x$, then $a(\tau, \tau') = b(\pi_{1*}\tau, \pi_{1*}\tau')$. The notation of § 2 still applies.

Proposition 3·2. Let f_1: $[a, b] \to B$ be C^1. For any $\lambda \in \Sigma$, the H-path (f_1, f_2): $[a, b] \to M$, for which $f_2(a) = \lambda$, is given by

$$\sigma \circ \{f_2(t) - \lambda\} = \int_a^t \delta\sigma(\dot{f}_1(t'))\, \mathrm{d}t' \quad (\sigma \in \Sigma_o^*). \tag{3·2}$$

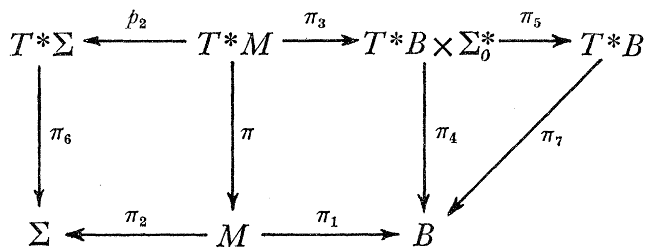
Here, $f_2(t) - \lambda$ denotes the translation $(\in \Sigma_o)$ from λ to $f_2(t)$.

Proof. Follows from the definition.

T^*M $(\sim T^*B \times \Sigma_o^* \times \Sigma)$, regarded as a principal bundle over $T^*B \times \Sigma_o^*$, carries a connexion $\hat{\Gamma}$ defined by the 1-form $\pi^*\gamma$. Horizontal paths (relative to $\hat{\Gamma}$) in T^*M can be determined as in proposition 3·2 from their projections into $T^*B \times \Sigma_o^*$. Because of the diffeomorphism

$$\zeta\colon T^*\Sigma \to \Sigma_o^* \times \Sigma, \tag{3·3}$$

we have the following commutative diagram in which all maps are projections.



(3·4)

Proposition 3·3. The spray Θ on T^*M projects under π_{3*} to a vector field Θ' on $T^*B \times \Sigma_o^*$ which is tangential to T^*B and is defined by the condition that

$$2\mathrm{d}\omega'(\phi', \Theta') = \phi' A' \tag{3·5}$$

for all ϕ' tangential to T^*B. Here, $\omega' \overset{\text{def}}{=} \pi_5^*\omega_1$, ω_1 being the fundamental 1-form for T^*B, and A' $(\in \mathscr{F}(T^*B \times \Sigma_o^*))$ is given by $A'(u_1, u_2) = \|u_1 + \delta u_2\|^2$. Moreover, Θ is the horizontal lift (relative to $\hat{\Gamma}$) of the field Θ'.

Proof. For any $\nu \in \Sigma$, set j_ν: $T^*B \times \Sigma_o^* \to T^*M$, where (using (3·3)) $j_\nu(u, v) = (u, \zeta^{-1}(v, \nu))$. Let $y = (y_1, y_2) \in T^*B \times \Sigma_o^*$, $\tau = (\tau_1, \tau_2) \in H_{\pi j_\nu y} M (\subset TM = TB \times T\Sigma)$. Then $\gamma(\tau) = 0$, so that $y_2 \mathrm{t}(\tau_2) = \delta y_2(\tau_1)$. Hence,

$$\begin{aligned}(j_\nu y)(\tau) &= y_1(\tau_1) + (\zeta^{-1}(y_2, \nu))(\tau_2) = y_1(\tau_1) + y_2(\mathrm{t}\tau_2) \\ &= (y_1 + \delta y_2)(\tau_1) = (y_1 + \delta y_2)(\pi_{1*}\tau) = b(\pi_{1*} a(j_\nu y), \pi_{1*}\tau).\end{aligned}$$

This holding for all τ, we have that $\pi_{1*} a(j_\nu y) = b(y_1 + \delta y_2)$. Accordingly,

$$A(j_\nu y) = \|\pi_{1*} a(j_\nu y)\|^2 = \|y_1 + \delta y_2\|^2.$$

This is independent of ν and thus defines a scalar A' on $T^*B \times \Sigma_o^*$ such that $A = \pi_3^* A'$. Hence, if $\psi \in TT^*M$ is tangential to Σ, we have $\psi A = 0$, so, by (2·5),

$$\mathrm{d}\omega(\psi, \Theta) = 0.$$

Now, $\omega = p_1^*\omega_1 + p_2^*\omega_2$, where p_1, p_2: $T^*M \to T^*B$, $T^*\Sigma$ are projections and ω_1, ω_2 are fundamental 1-forms on T^*B, $T^*\Sigma$, respectively. Since ψ is tangential to $T^*\Sigma$, we have $\mathrm{i}[\psi]\, \mathrm{d}(p_1^*\omega_1) = 0$, whence

$$\mathrm{d}(p_2^*\omega_2)(\psi, \Theta) = 0, \quad \text{all } \psi \text{ tangential to } \Sigma.$$

We write $p_2^*\Theta = \theta_1 + \theta_2$, where, by (3·3), θ_1, θ_2 may be chosen tangential to Σ_o^*, Σ, respectively. Then $p_{2*}\psi = \kappa_* \tilde{\psi}$, $\theta_2 = \kappa_* \tilde{\theta}_2$, where $\kappa\colon \Sigma \to T^*\Sigma$ is a constant section (constant 1-form on Σ). Accordingly, $d\kappa$ being zero, we have from proposition 2·1, (iii) for all ψ tangential to Σ,

$$d(p_2^*\omega_2)(\psi, \Theta) = d\omega_2(p_{2*}\psi, p_{2*}\Theta) = d\kappa(\tilde{\psi}, \theta_2) + d\omega_2(\kappa_*\tilde{\psi}, \theta_1)$$
$$= d\omega_2(\kappa_*\tilde{\psi}, \theta_1) = 0.$$

Again, by proposition 2·1, since θ_1 is vertical in $T^*\Sigma$, $i[\theta_1]\, d\omega_2 = \pi_6^*\tilde{\theta}_1$, where $\tilde{\theta}_1 \in T^*\Sigma$. So $d\omega_2(\kappa_*\psi, \theta_1) = \tilde{\theta}_1(\tilde{\phi}) = 0$ for arbitrary $\tilde{\psi}$. Hence, $\tilde{\theta}_1 = 0$, so that $\theta_1 = 0$ and $\pi_{3*}\Theta$ is tangential to T^*B. Finally, $p_1 = \pi_5\pi_3$, whence, combining (2·5) with the above results, we have for all $\phi' \in T(T^*B \times \Sigma_o^*)$ tangential to T^*B and all $\nu \in \Sigma$,

$$2\, d\omega(j_{\nu*}\phi', \Theta) = 2\, d(p_1^*\omega_1)(j_{\nu*}\phi', \Theta) = 2\, d\omega'(\pi_{3*}j_{\nu*}\phi', \pi_{3*}\Theta)$$
$$= 2\, d\omega'(\phi', \Theta') = (j_{\nu*}\phi')\, A = \phi' A',$$

giving (3·5). Observe that $d\omega_1$ is of maximal rank on T^*B, so that (3·5) does determine a C^∞ vector field Θ'. To complete the proof, note that Θ is horizontal relative to $\hat{\Gamma}$; in fact, $(\pi^*\gamma)(\Theta) = \gamma(\pi_*\Theta) = 0$, because geodesics in M are H-curves.

Notation 3·2. For any $P, Q \in B$, $\nu \in \Sigma_o$, let $\Omega(P, Q, \nu)$ denote the set of all C^0, piecewise C^1 maps $\phi\colon [a, b] \to B$, a, $b \in R$, such that

$$\phi(a) = P,\ \phi(b) = Q,\ \int_a^b \delta\sigma(\dot{\phi}(t))\, dt = \nu(\sigma),\ \text{all } \sigma \in \Sigma_o^*.$$

Problem A will be the isoperimetric problem of finding, in the set $\Omega(P, Q, \nu)$ (P, Q, ν fixed) a path ϕ for which the integral

$$\int_a^b \|\dot{\phi}(t)\|\, dt$$

is least. An *extremal* (of problem A) will be a C^1 map $\psi\colon R \to B$ such that for each $t \in R$ there exists $\epsilon > 0$ such that $\psi|[t-\epsilon, t+\epsilon]$ is minimizing in the set

$$\Omega\left(\psi(t-\epsilon), \psi(t+\epsilon), \sigma \to \int_{t-\epsilon}^{t+\epsilon} \delta\sigma(\dot{\psi}(t'))\, dt'\right).$$

Proposition 3·4. A C^1 map $R \to B$ is an extremal iff it is the projection into B of a geodesic of $\mathscr{P}\{B, \Sigma_o, \delta, b\}$.

Proof. Let $f\colon R \to M$ be a geodesic; by corollary 2·2, every sufficiently short sub-arc of f is minimizing. By proposition 3·2, and because H-paths in M have the same lengths as their projections into B, the map $\pi_1 f$ is an extremal of problem A. Conversely, an extremal in B lifts, via proposition 3·2, to an H-path in M having the minimizing property just stated. By the multiplier rule it is therefore a geodesic. (Equations (3·5) are, in fact, the Lagrange equations for problem A.)

Proposition 3·5. For every geodesic f of $\mathscr{P}\{B, \Sigma_o, \delta, b\}$ there is a one-dimensional subspace $\Sigma_{of}^* \subset \Sigma_o^*$ and a geodesic f' of $\mathscr{P}\{B, \Sigma_{of}, \delta, b\}$ such that the projections into B of f, f' coincide. (Thus, f is the lift into M of an extremal of problem A in which only one isoperimetric condition is imposed.)

Proof. Each Θ'-orbit, being tangential to T^*B, is of the form $t \to (f_1(t), f_2)$, where $f_2 \in \Sigma_o^*$ is independent of t. The result follows on taking for Σ_{of}^* the subspace of Σ_o^* spanned by f_2. In case $f_2 = 0$, f_1 is a geodesic of B, and Σ_{of}^* can be chosen arbitrarily.

NOTATION 3·3. Let $\mathfrak{g}$ denote the Lie algebra of the group of isometries of B and let $\mathfrak{g}.B$ denote the set of Killing fields on B. Set $Q \equiv \{\alpha \in \mathscr{F}_1 B \mid b(\alpha) \in \mathfrak{g}.B \text{ and } \mathrm{d}\alpha \equiv 0\}$.

Henceforth, we suppose that, for each $\sigma \in \Sigma_o^*$, $b(\delta\sigma) \in \mathfrak{g}.B$.

PROPOSITION 3·6. If $\dim Q = m$ and $x \in B$, then $T_x^* B = \bigcup_{\alpha \in Q} \alpha|_x$.

Proof. If $\alpha|_x = 0$ for some $\alpha \in Q$, then $\alpha \equiv 0$. For, B is connected (line 1, § 1) and each α is covariant constant, by Killing's equations and the condition $\mathrm{d}\alpha = 0$. By parallel propagation of α along curves issuing from x, we deduce $\alpha \equiv 0$, whence the result.

We require the following lemma which does not seem to be proved elsewhere.

LEMMA 3·1. If $\alpha \in \mathscr{F}_1 B$ is such that $b(\alpha)$ is a Killing field, the first-order prolongation, α', of $b(\alpha)$ in T^*B is α-related to $b(\alpha)$.

Proof. Recall α is a section $B \to T^*B$. For the proof, write α_2 instead of α, α_1 instead of $b(\alpha)$, and let α_1', α_2', respectively, denote the first-order prolongations of α_1 in TB, T^*B. We have to show that (suitably interpreted) $\alpha_{2*}(\alpha_1) = \alpha_2'$. Since B is complete (remark 1·1), α_1 generates a group of diffeomorphisms $\phi: B \times R \to B$, and we write $\phi_t(x)$ for $\phi(x,t)$. The following diagram commutes:

$$\begin{array}{ccc} TB & \xrightarrow{\ \phi_{t*}\ } & TB \\ \Big\uparrow{\scriptstyle b} & & \Big\uparrow{\scriptstyle b} \\ T^*B & \xrightarrow{\ \phi^*_{-t}\ } & T^*B \end{array}$$

To prove this, let $y \in T^*B$, $w \in T_z^*B$, where $z = \phi_t \pi y$. Since ϕ_t is an isometry,

$$w(\phi_{t*}\, b(y)) = (\phi_t^* w)\,(by) = b(y, \phi_t^* w)$$
$$= b(\phi^*_{-t} y, w) = w(b\phi^*_{-t} y).$$

Hence, y, w being arbitrary, $b\phi^*_{-t} = \phi_{t*}\, b$. The infinitesimal transformations of ϕ_{t*}, ϕ^*_{-t} being α_1', α_2', respectively, one deduces immediately that these latter are b-related: $\alpha_1' = b_*\, \alpha_2'$. If we now show that $\alpha_{1*}(\alpha_1) = \alpha_1'$, it will then follow that $\alpha_2' = (b^{-1})_* \alpha_1' = (b^{-1}\alpha_1)_* \alpha_1 = \alpha_{2*}(\alpha_1)$, as required. To prove that $\tau_1 = \tau_2$, where $\tau_1, \tau_2 \in T_y(TB)$, it suffices to show that

$$(\tilde{\pi}^*\beta)\,(\tau_1 - \tau_2) = 0$$

and that $\tau_1 \beta = \tau_2 \beta$ for all $\beta \in \mathscr{F}_1 B (\subset \mathscr{F}\, TB)$, where $\tilde{\pi}: TB \to B$. Obviously,

$$\pi_*\, \alpha_{1*}(\alpha_1) = \pi_*\, \alpha_1' (= \alpha_1),$$

so we need only establish that $\alpha_1' \beta = \alpha_{1*}(\alpha_1) \circ \beta$. Set $D = \mathrm{d}/\mathrm{d}t|_{t=0}$; at $\alpha_1|_x$ we have

$$\alpha_1' \beta = D[\beta\phi_{t*}\, \alpha_1] = D[\beta\alpha_1|_{\phi_t(x)}] = \alpha_1 \beta(\alpha_1) = \{\alpha_{1*}(\alpha_1)\}\,(\beta),$$

which completes the proof.

PROPOSITION 3·7. For any $\sigma \in \Sigma_o^*$ and $x \in B$, $t \to (\phi_{-2t}^* \delta\sigma(x), \sigma)$ is a Θ'-orbit. Hence,† the $2b(\delta\sigma)$-orbit $t \to \phi_{2t}(x)$ is an extremal of problem A.

Proof: Set l: $T^*B \to T^*B \times \Sigma_o^*$, $y \to (y, \sigma)$. Because $\pi\Psi_t = \phi_t \pi$, where $\Psi_t = \phi_{-t}^*$, if ω_1 is as in proposition 3·3, one checks that $\Psi_t^* \omega_1 = \omega_1$. Accordingly, $\mathscr{L}[\delta\sigma']\,\omega_1 = 0$, where $\delta\sigma'$ is the first order prolongation of $b(\delta\sigma)$ in T^*B, and so

$$\mathrm{i}[\delta\sigma']\,\mathrm{d}\omega_1 = -\mathrm{d}\{\omega_1(\delta\sigma')\}.$$

Thus, from (3·5), for any $y \in T_x^*B$, we have at (y, σ)

$$-2\,\mathrm{d}\omega'(\Theta' - 2l_* \,\delta\sigma', \phi') = \phi'(A' - 4\omega'(l_*\,\delta\sigma')), \tag{3·6}$$

where $$A' - 4\omega'(l_*\,\delta\sigma') = \|y + \delta\sigma(x)\|^2 - 4y(\delta\sigma(x)) = \|y - \delta\sigma(x)\|^2.$$

Since $\|y - \delta\sigma(x)\| = 0$ when $y = \delta\sigma(x)$ and is otherwise non-negative, $\mathrm{d}(A' - 4\omega'(l_*\delta\sigma'))$ vanishes at $(\delta\sigma(x), \sigma)$, whence, by (3·6), $\Theta' = 2l_*\,\delta\sigma'$—both vectors being tangential to T^*B. By lemma 3·1, the $l_*\,\delta\sigma'$-orbit through $(\delta\sigma(x), \sigma)$ remains in the section $\operatorname{im} l \circ \delta\sigma$, whence the $2l_*\,\delta\sigma'$-orbit $t \to (\phi_{-2t}^*\,\delta\sigma(x), \sigma)$ also does and is accordingly a Θ'-orbit.

PROPOSITION 3·8. If $\dim Q = m$ and $\delta\Sigma_o^* \cap Q = 0$, all extremals of problem A are trajectories of 1-parameter groups of isometries generated by Killing fields of the form $b(\nu)$, where $\nu \in \delta\Sigma_o^* \oplus Q$.

Proof. By proposition 3·4, an extremal is of the form $\pi_1\pi f$, where f is a Θ-orbit for the structure $\mathscr{P}\{B, \Sigma_o, \delta, b\}$. *A fortiori*, $\pi_1\pi f$ is the projection of a Θ-orbit, $\tilde{f}$, for the structure $\mathscr{P}(M, \tilde{H}, \tilde{a}) \equiv \mathscr{P}\{B, \tilde{\Sigma}_o, \tilde{\delta}, b\}$, where $\tilde{\Sigma}_o^* = \Sigma_o^* \oplus Q$, and $\tilde{\delta}\sigma = \delta\sigma$ if $\sigma \in \Sigma_o^*$, while $\tilde{\delta}\sigma = \frac{1}{2}\sigma$ if $\sigma \in Q$. In what follows, a tilde will signify objects associated with $\mathscr{P}(\tilde{M}, \tilde{H}, \tilde{a})$. Thus, $\tilde{M} = \tilde{B} \times \tilde{\Sigma}$, where $\tilde{B} = B$, $\tilde{b} = b$, and $\tilde{\Sigma}$ is affine space acted on by $\tilde{\Sigma}_o$. Choose $t_0 \in R$ and set $(y, \sigma) = \tilde{\pi}_3 f(t_0)$, $y \in T^*\tilde{B}$, $\sigma \in \tilde{\Sigma}_o^*$. Using proposition 3·6, let $\sigma^0 \in Q$ be such that $\sigma^0|_x = y - \tilde{\delta}\sigma|_x$, where $x = \tilde{\pi}_7 y$. Then (cf. (3·1)) $\sigma^0 \circ \gamma$ is a closed null 1-form on $\tilde{M}$. By proposition 2·9, $\tilde{f}$ projects to the same geodesic on M as the $\tilde{\Theta}$-orbit $\tilde{f}_1$ through the point $z = \tilde{f}(t_0) + \sigma^0 \circ \tilde{\gamma}|_\xi$, where $\xi = \tilde{\pi}\tilde{f}(t_0)$. Now, $\tilde{\pi}_3\tilde{f}_1$ is the $\tilde{\Theta}'$-orbit containing $\tilde{\pi}_3 z + (y - \tilde{\delta}\sigma^0|_x, \sigma + \sigma^0)$. By proposition 3·7, since $\tilde{\delta}(\sigma + \sigma^0)|_x = (\tilde{\delta}\sigma + \frac{1}{2}\sigma^0)|_x = y - \tilde{\delta}\sigma|_x$, the latter orbit projects under $\tilde{\pi}_4$ to an orbit of $\tilde{b}(2\tilde{\delta}\sigma + 2\tilde{\delta}\sigma^0)$ which, by construction, coincides with $\pi_1\pi f$.

Observe that, if $\delta\Sigma_o^* \cap Q \neq 0$, $\mathscr{P}\{B, \Sigma_o, \delta, b\}$ is not l.h.c. (definition 1·3).

4. A class of parabolic structures on R^n

Let $\mathscr{P}(M, H, a)$ be any parabolic space, let $o \in M$, and set δ: $N_o^* \to \mathscr{F}_1 H_o$, where, for $\mu \in N_o^*$, $\tau \in H_o$,

$$\delta\mu|_\tau = -\tfrac{1}{2}(\mathrm{i}[\tau]\,\mathrm{d}\mu)\,\mathrm{t_h} = -\tfrac{1}{2}\mathrm{t_h^*}\,\mathrm{i}[\tau]\,\mathrm{d}\mu. \tag{4·1}$$

Here, $\mathrm{t_h}$: $TH_o \to H_o$ takes a tangent vector to its isomorph. On H_o the tensor a defines a Euclidean metric, to be denoted by a_o. Clearly, $\mathscr{P}\{H_o, N_o, \delta, a_o\}$ is a parabolic structure in which, for each $\mu \in N_o^*$, $a_o(\delta\mu)$ is a Killing field representing an infinitesimal rotation of H_o about 0. We set $P_o \equiv H_o \oplus N_o$ (for notational reasons, $\oplus$ is preferable to $\times$),

$$H' = \{\eta \in TP_o | \gamma'(\eta) = 0\},$$

† That the $b(\delta\sigma)$-orbits are extremals of problem A was proved by Hermann (1962, §6). However, for the proof of proposition 3·8 we need to know the Θ'-orbits from which they arise. Although an extension of Hermann's calculation yields this result, the following proof is more in line with this paper.

where γ' is the 1-form on P_o with values in N_o given, as in (3·1), by

$$\left.\begin{array}{l}\sigma \circ \gamma'(\eta) = \sigma \mathrm{t_n}(\pi_{\mathrm{n}*}\eta) - \delta\sigma(\pi_{\mathrm{h}*}\eta),\ \sigma \in N_o^*,\\ N_o \xleftarrow{\pi_\mathrm{n}} P_o \xrightarrow{\pi_\mathrm{h}} H_o.\end{array}\right\} \quad (4·2)$$

On P_o we have the parabolic metric a_o', given by $a_o'(\eta_1, \eta_2) = a_o(\mathrm{t_h}\,\pi_{\mathrm{h}*}\eta_1, \mathrm{t_h}\,\pi_{\mathrm{h}*}\eta_2)$ whenever $\eta_1, \eta_2 \in H_x' P_o$, some $x \in P_o$. Thus, $\mathscr{P}\{H_o, N_o, \delta, a_o\} \equiv \mathscr{P}(P_o, H', a_o')$. We write $|\xi| = \surd a_o'(\xi, \xi)$ if $\xi \in H'$ or if $\xi \in T^*P_o$, and $\|\eta\| = \surd a_o(\eta, \eta)$ if $\eta \in TH_o$.

For the case $B = H_o$ in notation 3·3, Q is the set of constant 1-forms on H_o and dim $Q = m$. So, by propositions 3·7 and 3·8, all extremals in H_o are orbits of 1-parameter groups of isometries and vice versa. Let $(y_1, y_2) \in T_x^* H_o \times N_o^*$; by the proof of proposition 3·8 the Θ'-orbit through (y_1, y_2) projects to an orbit of $a_o(2\tilde{\delta} y_1 + 2\tilde{\delta}\sigma^0) = a_o(2\delta y_2 + \sigma^0)$, where σ^0 is the constant 1-form on H_o such that $\sigma^0|_x = y_1 - \tilde{\delta} y_2|_x$.

NOTATION 4·1. In the rest of § 4, the symbols $\mathrm{t_h}$, $\mathrm{t_n}$ will be suppressed, and no notational distinction will be made between tangent vectors to H_o, N_o and their isomorphs. We write

$$\Delta\mu = a_o \circ \mathrm{d}\mu, \quad \mu \in N_o^*, \quad (4·3)$$

so that $\Delta\mu$ is an endomorphism $H_o \to H_o$ (iso if H has co-rank m) and $\Delta\mu(\tau) = 2a_o\,\delta\mu|_\tau$. We set up a matrix type of notation whereby for any $\tau \in H_o$, $\tau^t \in H_o^*$ is the transpose, given by $\tau = a_o(\tau^t)$, $(\tau^t)^t = \tau$. For a linear operator α: $H_o \to H_o$, define α^t by $a_o(\alpha^t x, y) = a_o(x, \alpha y) = x^t \alpha y$. Note that $\Delta\mu^t = -\Delta\mu$. The exponential map $P_o^* \to P_o$ for $\mathscr{P}(P_o, H', a_o')$ at 0 will be written e_o'. The extremal $r \to x(r)$ in H_o, which is the projection of the Θ'-orbit through

$$(y^t, \mu) \in T_o^* H_o \times N_o^*$$

is given by
$$\mathrm{d}x/\mathrm{d}r = \alpha x + y, \quad \alpha = \Delta\mu, \quad x(0) = 0,$$

since $\tilde{\delta}\mu|_o = 0$. *Assuming henceforth that H has co-rank m* (so that every $\xi \in H$ is variationally normal—cf. definition 2·7), we have, by integration,

$$x = \begin{cases}(\mathrm{e}^{\alpha r} - 1)\,\alpha^{-1} y & (\mu \neq 0),\\ ry & (\mu = 0).\end{cases} \quad (4·4)$$

If $\|y\| = \surd(y^t y) = 1$, then $\|\dot{x}(r)\| \equiv 1$, so that r represents arc-length measured from 0.

NOTATION 4·2. Define $H_o^c = \mathfrak{C} \otimes H_o$, i.e. H_o with complex numbers as ground field. For $\mu \in N_o^*$, set $\rho(\mu) = \max(\rho_1, \ldots, \rho_m)$ (> 0) where $\mathrm{i}\rho_1, \ldots, \mathrm{i}\rho_m$, $\mathrm{i} = \surd -1$, are the eigenvalues of the operator $\Delta^c\mu$: $H_o^c \to H_o^c$, $z \otimes v \to z \otimes \Delta\mu(v)$. Define

$$N_{o1}^* = \{\mu \in N_o^* \,|\, \rho(\mu) < 2\Pi\}, \quad P_{o1}^* = H_o^* \oplus N_{o1}^* = \{y \in P_o^* \,|\, \rho(\pi_\mathrm{n}' y) < 2\Pi\},$$

where π_n': $P_o^* \to N_o^*$.

PROPOSITION 4·1. N_{o1}^* is convex.

Proof. Any $\mu \in N_{o1}^*$ is characterized by the condition

$$\mathrm{i}\,\mathrm{d}^c\mu(\bar{\xi}, \xi) < 2\Pi \text{ for all } \xi \in H_o^c \text{ such that } a_o^c(\xi, \bar{\xi}) = 1,$$

where $\mathrm{d}^c\mu$ (exterior derivative of μ), a_o^c have their obvious meanings. The convexity of N_{o1}^* follows immediately.

THEOREM 4·1. For each $p \in P^*_{01}$ the geodesic arc $f: I \to P_o$, $r \to e'_o(rp)$, has smaller energy than any other H'-arc joining 0 to $f(1)$.

Proof. Set $p = y^t + \mu$, $y \in H_o$, $\mu \in N^*_{01}$. If $\mu = 0$, $r \to f(r) = yr$ is a geodesic for H_o with its Euclidean metric. There is a compatible Euclidean metric on $H_o \oplus N_o$ which admits f, so f has the property required.

Since on f, r is proportional to J_1 length, it is sufficient (cf. proposition 1·1) to prove that, for $\mu \neq 0$, any H'-arc, f', which joins 0 to $f(1)$, has $J_1(f') > J_1(f) = |p|$. Let N^*_μ be the subspace of N^*_o spanned by μ. By considering the projection of f into H_o, using propositions 3·4 and 3·5, one sees that it suffices to consider the case where $N^*_o (= N^*_\mu)$ is one-dimensional. Hence, if f does not have the property of the theorem, there is an H'-arc $f': I \to P_o$ for which $f'(0) = 0$, $f'(1) = f(1)$ and $J_1(f') \leqslant J_1(f)$. By proposition 2·15, assume f' to be absolutely minimizing, and parametrized proportionally to arc length, i.e. of the form $s \to e'_o(sq)$, where $q = z^t + v\mu$, $z \in H_o$, $v \in R$, $s \in I$.

For $u \in N_o$, $\mu(u)$ is a convenient coordinate. By (3·2), (4·4), f is given by

$$\pi_h f(s) = (e^{\alpha s} - 1)\,\alpha^{-1} y, \quad \alpha = \Delta\mu, \tag{4·5}$$

$$\mu\{\pi_n f(s)\} = \int_0^s \delta\mu(\pi_{h*}\dot{f}(r))\,dr, \quad \pi_{h*}\dot{f}(r) = e^{\alpha r} y.$$

Hence, from (4·1), with $\tau = \pi_h f(r)$, and (4·3),

$$\begin{aligned} 2\mu\{\pi_n f(s)\} &= \int_0^s (e^{\alpha r} y)^t \alpha (e^{\alpha r} - 1)\,\alpha^{-1} y\,dr \\ &= \int_0^s y^t (1 - e^{-\alpha r})\,y\,dr \\ &= y^t (s - \alpha^{-1} \sinh \alpha s)\,y. \end{aligned} \tag{4·6}$$

Accordingly, from (4·5) $\quad 4\mu\{\pi_n f(1)\} = \chi(1),$

where $$\chi(\theta) \equiv (\alpha x)^t (1 - \cosh\theta\alpha)^{-1} (\theta - \alpha^{-1}\sinh\theta\alpha)\,(\alpha x), \tag{4·7}$$

and $x = \pi_h f(1)$.

Choose a basis in H^c_o. With respect to this basis, the operator $1 - \cosh\alpha^c t$, where $\alpha^c = \Delta^c\mu$ (notation 4·2), has a reciprocal for all $t \in [0, 1]$. One sees this by writing $\cosh\alpha^c t = \cos i\alpha^c t$ and diagonalizing the Hermitian matrix $i\alpha^c$ by a unitary matrix:

$$i\alpha^c = \tilde{U}\,\mathrm{diag}\,(\rho_1, \ldots, \rho_m)\,U, \quad (\tilde{\ }) = \text{Hermitian conjugate}, \tag{4·8}$$

where $|\rho_j| < 2\Pi$, each j. Here, $\rho_{r+i} = -\rho_i$, $i = 1, \ldots, r$; $r = \frac{1}{2}m$.

The geodesic f' has, for some $v \in R^+$, $z \in H_o$, the representation $s \to f'(s)$, where

$$\left.\begin{aligned} \pi_h f'(s) &= v^{-1}(e^{\alpha v s} - 1)\,\alpha^{-1} z \\ 2\mu\{\pi_n f'(s)\} &= v^{-1} z^t (s - v^{-1}\alpha^{-1}\sinh\alpha v s)\,z. \end{aligned}\right\} \tag{4·9}$$

However, since $f'(1) = f(1)$, $$x = v^{-1}(e^{v\alpha} - 1)\,\alpha^{-1} z, \tag{4·10}$$

$$4\mu\{\pi_n f'(1)\} = \chi(v) = \chi(1) = 4\mu\{\pi_n f(1)\}, \tag{4·11}$$

provided that $1 - \cosh v\alpha$ is non-singular. We show below that $\chi(\theta)$ is an increasing function of θ in the range $[0, \lambda)$, where $\lambda = 2\Pi/\rho(\mu) > 1$. Then (4·11) will imply either $v = 1$ or $v \geqslant \lambda$.

If $v = 1$, we have the contradiction $f = f'$. The possibility $v > \lambda$ is eliminated because, by lemma 4·1 below, the point $s = v^{-1}\lambda$ is conjugate for f', whence, by proposition 2·13, f' is not relatively minimizing. Finally, consider the case $v = \lambda$. By means of (4·8) one verifies that, as $\theta \to \lambda$,

$$(1-\cosh \alpha\lambda)\,(1-\cosh \alpha\theta)^{-1} \to \tilde{U}\nabla U,$$

where ∇ ($= \operatorname{diag}(\nabla_1, \ldots, \nabla_m)$) has only 0's (at least 2) and 1's on the diagonal. Moreover, by (4·8),

$$1-\theta^{-1}\alpha^{-1}\sinh \theta\alpha = \tilde{U}\operatorname{diag}(\delta_1, \ldots, \delta_m)\,U,$$

$$\delta_i = 1-\rho_i\,\theta^{-1}\sin \rho_i\,\theta,\ i = 1, \ldots, m.$$

Hence, inserting (4·10) (with $v = \lambda$) into (4·7) and letting θ tend to λ from below, we obtain

$$\chi(\lambda-) = 2\lambda^{-1}\sum_{j=1}^{m} |Uz|_j^2\,\nabla_j(1-\lambda^{-1}\rho_j^{-1}\sin \lambda\rho_j), \tag{4·12}$$

where $|Uz|_j$ is the modulus of the jth component of Uz. A similar analysis using (4·9) shows that $4\mu\{\pi_n f'(1)\}$ has the same expression as $\chi(\lambda-)$ except that each ∇_i is replaced by 1. But every term in the sum (4·12) is non-negative, so we reach the contradiction

$$4\mu\{\pi_n f(1)\} = \chi(1) < \chi(\lambda-) \leqslant 4\mu\{\pi_n f'(1)\}.$$

To complete the proof we show that $\chi(\theta)$ increases in the range $[0, \lambda)$. Differentiation of (4·7) gives

$$\chi'(\theta) = 2(\alpha x)^t(1-\cosh \theta\alpha)^{-1}\,(1-\tfrac{1}{2}\theta\alpha \cosh \tfrac{1}{2}\alpha)\,(\alpha x),$$

and this is positive for $\theta \in [0, \lambda)$.

Corollary 4·1. $P_{o1}^* \setminus N_o^*$ contains no conjugate points.

Corollary 4·2. $e_o'|(P_{o1}^* \setminus N_o^*)$ is $1-1$.

Set π_h', π_n': $T_0^*P_o \to H_o^*, N_o^*$, respectively. (We shall not in this section distinguish notationally between $T_0^*P_o$ and P_o^*.)

Lemma 4·1. If $\dim N_o^* = 1$, the set $\{p \in T_0^*P_o | \rho(\pi_n' p) = 2\Pi\}$ is conjugate.

Proof. Write $p = y^t + \mu$, where $y \in H_o, \mu \in N_o^*, \rho(\mu) = 2\Pi$. We have

$$\left.\begin{aligned} \pi_h\,e_o'p &= (e^{\alpha}-1)\,\alpha^{-1}y,\ \alpha = \Delta\mu, \\ 2\mu(\pi_n\,e_o'p) &= y^t(1-\alpha^{-1}\sinh \alpha)\,y. \end{aligned}\right\}$$

There is a non-zero eigenvector $\eta = \eta_1 + i\eta_2 \in H_0^c(\eta_1, \eta_2 \in H_o)$ such that $\alpha\eta = 2\pi i\eta$, namely

$$\alpha\eta_1 = 2\Pi\eta_2, \quad \alpha\eta_2 = -2\Pi\eta_1.$$

Note that this implies $\eta_1 \wedge \eta_2 \neq 0$. Consider η_1^t, η_2^t as vectors in $T_p(P_o^*)$ tangential to H_o^*. Since $e^{\alpha}\eta = \eta$,

$$\left.\begin{aligned} \pi_h\,e_{o*}'\,\eta_i^t &= (e^{\alpha}-1)\,\alpha^{-1}\eta_i = 0, \\ \mu(\pi_{n*}\,e_{o*}'\,\eta_i^t) &= y^t(\eta_i) \end{aligned}\right\} \quad (i = 1, 2).$$

Hence,

$$e_{o*}'(y^t(\eta_2)\,\eta_1^t - y^t(\eta_1)\,\eta_2^t) = 0,$$

so that p is conjugate for e_o'.

If $\tau \in T_y P_o^*$, $e_{o*}'\,\tau (\in TP_o)$ is isomorphic to an element of P_o, and hence to an element of $T_y^*P_o^*$. One can therefore associate with e_{o*}' a tensor $\mathscr{E} \in \mathscr{F}_2 P_o^*$ whose components, in terms of a basis for P_o^*, are those of the Jacobian matrix of e_o' at y. The following result will be important in §7 for investigating the Jacobian of the exponential map e_o in an arbitrary parabolic space.

Proposition 4·2. There is a (convex) neighbourhood $N_o^*(\epsilon)$ of the origin in N_o^* such that the Jacobian $\mathscr{E}$ of e_o' is positive-definite on the set $\mathscr{B}_o' \setminus N_o^*$, where $\mathscr{B}_o' = H_o^* \oplus N_o^*(\epsilon)$. Moreover, for any non-zero $\tau (\in T_y P_o^*)$ tangential to H_o^* at any $y \in N_o^*(\epsilon)$ we have $\mathscr{E}(\tau, \tau) > 0$.

Proof. First, we prove that $\mathscr{E}|_y$ is positive-definite for all y near the unit sphere in H_o^*. For $y \in P_o^*$, $z \in P_o$, write $y_h = \pi_h' y$, $y_n = \pi_n' y$, $z_h = \pi_h z$, $z_n = \pi_n z$. By (3·2) and (4·4), if $y_n \neq 0$,

$$(e_o' y)_h = (e^\alpha - 1)\,\alpha^{-1} y_h^t, \quad \alpha = \Delta y_n, \tag{4·13}$$

$$2\sigma(e_o' y)_n = \int_0^1 y_h\, e^{-\alpha s} \Delta\sigma (e^{\alpha s} - 1)\,\alpha^{-1} y_h^t\, ds,\ \sigma \in N_o^*. \tag{4·14}$$

The formula for the case $y_n = 0$ can be found from these by continuity. Let $U \in P_o^*$ and let $\tilde{u}$ be the constant tangent field on P_o^* isomorphic to u. Then $\tilde{u}$ operates on $e_{o*}'(\tilde{u})$ as already indicated, and we have

$$\mathscr{E}(\tilde{u}, \tilde{u}) = \tilde{u} \circ e_{o*}'(\tilde{u}) = \tilde{u}_h \circ e_{o*}'(\tilde{u}_h) + \tilde{u}_h \circ e_{o*}'(\tilde{u}_n) + \tilde{u}_n \circ e_{o*}'(\tilde{u}_h) + \tilde{u}_n \circ e_{o*}'(\tilde{u}_n).$$

By the linearity of (4·13) in y_h, $e_{o*}'(\tilde{u}_h|_y)$ is the isomorph of $e_o'(u_h + y_n)$, so that

$$\tilde{u}_h \circ e_{o*}'(\tilde{u}_h) = u_h \circ e_o'(u_h + y_n) = u_h (e^\alpha - 1)\,\alpha^{-1} u_h^t. \tag{4·15}$$

As $\alpha = \Delta y_n \to 0$, this tends to $u_h u_h^t$. Further calculations using (4·13), (4·14) yield

$$\mathscr{E}(\tilde{u}, \tilde{u})|_{y_n = 0} = \|(u_h + \tfrac{1}{4}\Delta u_n \circ y_h^t\|^2 + \tfrac{1}{48}\|\Delta u_n \circ y_h^t\|^2, \tag{4·16}$$

which, if $y_h \neq 0$, is positive for all $u \neq 0$, by definition 1·4.

Thus, there is an $\epsilon \in (0, \Pi)$ such that $\mathscr{E}$ is positive-definite on the set

$$Y \equiv \{y \in P_o^* \mid \|y_h\| = 1,\ \rho(y_n) < \epsilon\}.$$

Inspection of (4·16) reveals that, under the transformation

$$y_h \to \lambda y_h, \quad \tilde{u}_h \to \lambda \tilde{u}_h, \quad \lambda \in R,$$

$\mathscr{E}(\tilde{u}, \tilde{u})|_y$ is multiplied by λ^2. Hence,

$$\mathscr{E}(\tilde{u}, \tilde{u})|_y = \lambda^{-2} \mathscr{E}(\lambda \tilde{u}_h + \tilde{u}_n, \lambda \tilde{u}_h + \tilde{u}_n)|_{\lambda y_h + y_n}.$$

Define $N_o^*(\epsilon) = \{\mu \in N_o^* \mid \rho(\mu) < \epsilon\}$. Then $y \in \mathscr{B}_o' \setminus N_o^*$ implies that $\lambda y_h + y_n \in Y$, where $\lambda = \|y_h\|^{-1}$, giving $\mathscr{E}(\tilde{u}, \tilde{u})|_y > 0$ for all $\tilde{u}$.

Finally, by (4·15),

$$\mathscr{E}(\tilde{u}_h, \tilde{u}_h) = u_h (e^\alpha - 1)\,\alpha^{-1} u_h^t.$$

From notations 4·2 and a simple calculation using (4·8), one sees that this is positive whenever $y \in N_o^*(\epsilon)$ and $u_h \neq 0$. The set $N_o^*(\epsilon)$, being similar to N_{o1}^*, is convex, and hence $\mathscr{B}_o'$ has the properties stated.

Notation 4·3. Choose a basis $(\eta'^1, \ldots, \eta'^n)$ for P_o^* such that $(\eta'^1, \ldots, \eta'^m)$, $(\eta'^{m+1}, \ldots, \eta'^n)$ span H_o^*, N_o^*, respectively, and let $(\eta_1', \ldots, \eta_n')$ be the dual basis for P_o. Write

$$\chi': R^n \to P_o^*,\ (y_1, \ldots, y_n) \to y_r \eta'^r;\ \psi': P_o \to R^n,\ z^r \eta_r' \to (z^1, \ldots, z^n);\ \epsilon_o' = (\epsilon_o'^1, \ldots, \epsilon_o'^n) = \psi' e_o' \chi'.$$

In §7 we shall want proposition 4·2 in the following coordinate form.

PROPOSITION 4·2*a*. With $\mathscr{B}'_o$ as in proposition 4·2, the Jacobian matrix function

$$(J'^{ij}) \equiv (\partial \epsilon'^{i}_{o}(y_1, \ldots, y_n)/\partial y_j)$$

is positive-definite on the set $\chi'^{-1}(\mathscr{B}'_o \setminus N^*_o)$, while the matrix $(J'^{\alpha\beta})$ $(\alpha, \beta = 1, \ldots, m)$ is positive-definite on $\chi'^{-1}(\mathscr{B}'_o \cap N^*_o)$.

DEFINITION 4·1. $Z \equiv \{q \in P^*_o \setminus N^*_o \,|\, \rho(\pi'_n q) > 2\nu\Pi\}$ where $\nu = [\frac{1}{2}(n-m)]$.

PROPOSITION 4·3. If $p \in Z$, the geodesic arc $I \to M$, $t \to e'_o(tp)$ is not relatively minimizing.

Proof. Let $k = y^t + \mu = \{2\nu\Pi/\rho(\pi'_n p)\}p$, where $y \in H_o$, $\mu \in N^*_o$, so that $\rho(\mu) = 2\nu\Pi$; also, write $I_\lambda = [(\lambda-1)/\nu, \lambda/\nu]$, $\lambda = 1, \ldots, \nu$. Define $f\colon I \times R \to M$, where, for $u \in I_\lambda$,

$$\pi_h f(u, v) = (e^{\alpha u} - 1)\,\alpha^{-1}\xi, \; \xi = y + a_\lambda(v)\,\eta_1 + b_\lambda(v)\,\eta_2, \; \alpha = \Delta\mu,$$

$$a_\lambda(v) \equiv -y^t(\eta_1)\,(1-\cos c_\lambda v) + y^t(\eta_2)\sin c_\lambda v,$$

$$b_\lambda(v) \equiv -y^t(\eta_2)\,(1-\cos c_\lambda v) - y^t(\eta_1)\sin c_\lambda v.$$

Here, η_1, η_2 $(\in H_o)$ are such that

$$\alpha\eta_1 = 2\nu\Pi\eta_2, \quad \alpha\eta_2 = -2\nu\Pi\eta_1, \quad \|\eta_1\| = \|\eta_2\| = 1,$$

and $c_1, \ldots, c_\nu$ are constants whose values will be assigned presently. Finally, $\pi_n f$ is to be such that, for each $v \in R$, $f_v\colon I \to M$, $u \to f(u, v)$, is an H-arc for which $f_v(0) = 0$. Check that, for all v,

$$J(f_v) = |k| \; (= \|y\|), \tag{4·17}$$

and that f is a 1-parameter family of H-paths (§2). However, $u \to f_*(\partial/\partial v|_{(u,0)})$ is not a variation by our definition unless $f_*(\partial/\partial v|_{(1,0)}) = 0$. We now show that $c_1, \ldots, c_\nu$ can be chosen so that this condition is verified; it will then follow from (4·17) that the corresponding second variation is zero, and hence that either k is conjugate, or that zero is not the least possible value for a second variation of the geodesic arc f_0. In either case the proposition is established.

From (3·2), for any $\sigma \in N^*_o$,

$$\sigma\{\pi_n f_v(1)\} = \int_0^1 \xi^t e^{-\alpha u} \Delta\sigma (e^{\alpha u} - 1)\,\alpha^{-1}\xi \, du.$$

At $v = 0$, $(d/dv)\,\sigma\{\pi_n f_v(1)\}$ is of the form

$$\sum_{\lambda=1}^{\nu} c_\lambda \mathscr{I}_\lambda(\sigma), \; \mathscr{I}_1, \ldots, \mathscr{I}_\nu \in N_o.$$

Since $2\nu > n - m = \dim N_o$, the $\mathscr{I}_\lambda$'s are linearly dependent, whence the result.

5. THE MAP $\hat{e}_o$

In this section $\mathscr{P}(M, H, a)$ is an arbitrary parabolic space.

DEFINITION 5·1. $\tilde{P}_o$ is the set of equivalence classes of C^2 paths $f\colon I \to M$ such that $f(0) = o \in M$, $\dot{f}(0) \in H_o$, two such paths, f_1, f_2, being equivalent if, for all $\mu \in \mathscr{N}_o$,

$$(1)\;\; \dot{f}_1(0) = \dot{f}_2(0); \quad (2)\;\; (d/dt)\,[\mu(\dot{f}_1(t)) - \mu(\dot{f}_2(t))]|_{t=0} = 0.$$

NOTATION 5·1. d_t, d^2_t, respectively, will denote d/dt, d^2/dt^2, with evaluation at $t = 0$.

DEFINITION 5·2. There is a bijection $\phi_o: \tilde{P}_o \to P_o$ defined as follows. Let $[f] \in \tilde{P}_o$, where $\dot{f}(0) = f_1$. If $\mu, \mu' \in \mathcal{N}_o$ are such that $(\mu - \mu')|_o = 0$, and if $\mu^1, \ldots, \mu^\nu$ is a basis for $\mathcal{N}_o$, we have $\mu - \mu' = c_\sigma . \mu^\sigma$, $c_\sigma \in \mathcal{F}$, $c_\sigma(o) = 0$, $\sigma = 1, \ldots, \nu$. Hence,

$$d_t[(\mu-\mu') \circ \dot{f}(t)] = d_t[c_\sigma f(t)] . \mu^\sigma \circ \dot{f}(0) + c_\sigma(o) . d_t[\mu^\sigma \circ \dot{f}(t)] = 0.$$

The number $d_t[\mu \dot{f}(t)]$ is therefore determined by the values of $\mu|_o \in N_o^*$ and $[f] \in \tilde{P}_o$. Since the map $f_2: N_o^* \to R$ given by $\mu \to \frac{1}{2} d_t[\mu \dot{f}(t)]$ is linear, we have $f_2 \in N_o$. Hence, ϕ_o, defined by $[f] \to f_1 + f_2$, is $1-1$. On the other hand, given $f_1 \in H_o$, $f_2 \in N_o$, one readily constructs, by means of a chart at o, a C^2 path $f: I \to M$ such that $f(0) = o$, $\dot{f}(0) = f_1$, $\phi_o[f] = f_1 + f_2$.

DEFINITION 5·3. Denote by $'N_o^*$ the set of equivalence classes of functions $\chi \in \mathcal{F}_o$ for which $\chi(o) = 0$, $d\chi|_o \in N_o^*$ and $d_t^2(\chi f) = 0$ for all C^2 H-paths f such that $f(0) = o$. Two such functions, χ_1, χ_2, are equivalent if $d(\chi_1 - \chi_2)|_o = 0$. For each $\mu \in N_o^*$, denote by $'\mu$ the unique element of $'N_o^*$ for which $'\mu \subset \mu$ (recall $\mu \subset \mathcal{F}_o$). $'N_o^*$ has a vector space structure isomorphic to N_o^*.

PROPOSITION 5·1. For each $\mu \in N_o^*$, $'\mu \neq \varnothing$. If $[f] \in \tilde{P}_o$ and $\psi \in {'\mu}$, $d_t^2 \psi f(t) = d_t \mu \dot{f}(t)$.

Proof. Let $\{x^i\}$ be coordinate functions at o such that $x^i(o) = 0$ and set $d\psi = \psi_i \, dx^i$, $\mu = \mu_i dx^i$, where $\mu_i|_o = \psi_i|_o$. The first assertion is established by the fact that $\mu_i x^i - \mu_{ij} x^i x^j \in {'\mu}$, where $d\mu_i = 2\mu_{ij} \, dx^j$.

Next, let f_1 be an H-path such that $\dot{f}_1(0) = \dot{f}(0)$; writing

$$f^i(t) = f(t) \circ x^i, \quad f_1^i(t) = f_1(t) \circ x^i,$$

we have

$$\begin{aligned} 0 = d_t^2 \psi f_1(t) &= d_t\{\psi_i f_1(t) . \dot{f}_1^i(t)\} \\ &= \dot{f}_1(0) \circ \psi_i . \dot{f}_1^i(0) + \psi_i(o) . d_t \dot{f}_1^i(t) \\ &= \dot{f}(0) \circ \psi_i . \dot{f}^i(0) + \psi_i(o) . d_t \dot{f}_1^i(t); \end{aligned} \tag{5·1}$$

$$\begin{aligned} 0 = d_t \mu \dot{f}_1(t) &= \dot{f}_1(0) \circ \mu_i . \dot{f}_1^i(0) + \mu_i(o) . d_t \dot{f}_1^i(t) \\ &= \dot{f}(0) \circ \mu_i . \dot{f}^i(0) + \mu_i(o) . d_t \dot{f}_1^i(t). \end{aligned} \tag{5·2}$$

Comparison of (5·1) and (5·2) shows that

$$\dot{f}(0) \circ \psi_i . \dot{f}^i(0) = \dot{f}(0) \circ \mu_i . \dot{f}^i(0),$$

whence

$$d_t^2 \psi f(t) = \dot{f}(0) \circ \mu_i . \dot{f}^i(0) + \mu_i(o) . d_t \dot{f}^i(0) = d_t \mu \dot{f}(t).$$

NOTATION 5·2. Let M_1, M_2 be manifolds and let $i: M_1 \to M_2$ be a regular embedding. We define the *normal bundle* $(M_1, M_2)^\perp$ of M_1 in M_2 to be the quotient $i^{-1}(TM_2)/TM_1$, where $i^{-1}(TM_2)$ is the induced bundle over M_1 (Steenrod 1951, p. 47). Let

$$i_o: N_o^* \to T_o^*, \quad i_o': N_o^* \to T^*, \quad i_1: N_o^* \to N^*, \quad i_2: N^* \to T^*$$

be inclusions and, as always, let $\pi: T^* \to M$. O_M will denote the zero vector bundle over M. The following are vector bundles over N^*

$$\mathcal{M} \equiv i_2^{-1}(TT^* \cap \pi_*^{-1} H) \subset TT^*, \quad \mathcal{N} \equiv \mathcal{M}/(TN^* \cap \pi_*^{-1} O_M),$$

$$\hat{M} \equiv i_2^{-1}(TT^*) \cap \pi_*^{-1} O_M \subset \mathcal{M}, \quad \hat{N} \equiv \hat{M}/(TN^* \cap \pi_*^{-1} O_M) \subset \mathcal{N}.$$

The following are vector bundles over N_o^*, where $o \in M$ is arbitrary

$$\hat{M}^o \equiv i_1^{-1} \hat{M} \subset \hat{M}, \quad \hat{N}^o \equiv i_1^{-1} \hat{N} \equiv (N_o^*, T_o^*)^\perp \subset \hat{N},$$

$$\mathcal{M}^o \equiv i_1^{-1} \mathcal{M} \subset \mathcal{M}, \quad \mathcal{N}^o \equiv i_1^{-1} \mathcal{N} \subset \mathcal{N}.$$

There are natural surjections

$$\hat{\pi}: \hat{M} \to \hat{N}, \quad \check{\pi}: \mathcal{M} \to \mathcal{N}, \quad \hat{\pi}_o: \hat{M}^o \to \hat{N}^o, \quad \check{\pi}_o: \mathcal{M}^o \to \mathcal{N}^o.$$

Define $P(=PM) = H \oplus N$, $P^* = H^* \oplus N^*$. There are diffeomorphisms

$$j: \hat{N} \to P^*, \quad j_o: \hat{N}^o \to P^*_o,$$

where $j_o = j|\hat{N}^o$. To define j, let $\nu \in \hat{N}$ belong to the fibre over $\nu_2 \in N^*_o$, $o \in M$. Then ν is a subspace (coset) of $T_{\nu_2}(T^*_o)$ whose isomorph in T^*_o is an element $\nu_1 \in H^*_o (\approx T^*_o/N^*_o)$. Define $j(\nu) = \nu_1 + \nu_2$. Further, define $p_o \equiv i^*_h j_o: \hat{N}^0 \to H^*_o$, where $i_h: H_o \subset P_o$. If $\kappa \in \mathscr{F}_r M$, $r = 0, 1, \ldots,$ we often write ${}^e\kappa$ instead of $e^*_o \kappa$. Let $\chi \in \mathscr{F} X$, where X is a manifold, and let $U \subset X$ be the critical set for χ. Then $\mathscr{H}[\chi] \in \mathscr{F}_2 X|U$ will denote the Hessian of χ on U; it is defined for $x \in U$, τ, $\tau' \in T_x X$ by $2\mathscr{H}[\chi](\tau, \tau') = \tau\tau'(\chi)$. If $\mu \in N^*_o$, $\psi \in {}'\mu$, then ${}^e\psi$ is critical on N^*_o (by (*a*) below), and so $\mathscr{H}[{}^e\psi]: \hat{M}^o \to R$ is defined. From proposition 5·1 and the symmetry of the Hessian, one deduces that $\mathscr{H}[{}^e\psi]$ is the same for all $\psi \in {}'\mu$ so we define $\mathscr{H}[{}^e\mu] \equiv \mathscr{H}[{}^e\psi]$, where $\psi \in {}'\mu$. Observe that, for each $\tau \in \hat{M}^o$, $\mu \to \mathscr{H}[{}^e\mu](\tau, \tau)$ is linear, so that $\mathscr{H}[{}^e ?](\tau, \tau) \in N_o$.

Proposition 5·2. The exponential map e induces a fibre-preserving map $\hat{e}: P^* \to P$.

Proof. We construct, for each $o \in M$ a map $\hat{e}_o: P^*_o \to P_o$, and then define $\hat{e}$ by $\hat{e}|P^*_o = \hat{e}_o$. Since $\hat{\pi}_o$ is onto, given $q \in P^*_o$, we can choose $\hat{q} \in \hat{M}^o$ such that $q = j_o \hat{\pi}_o \hat{q}$. Write $q = q_1 + q_2$, where $q_1 \in H^*_o$, $q_2 \in N^*_o$. Construct a C^2 path $f_q: I \to T^*_o$ such that $f_q(0) = q_2$, $\dot{f}_q(0) = \hat{q}$. The C^2 path $e_o f_q: I \to M$ has, as we show in (*a*) below, its tangent vector at o in H_o, so it defines, by definition 5·1, a class $[e_o f_q] \in \tilde{P}_o$. In (*b*) we shall prove that $[e_o f_q]$ is independent of the choice of $\hat{q}$ representing q, and of f_q tangential to $\hat{q}$ at q_2. Using definition 5·2, we can therefore define $\hat{e}_o$ by $q \to \phi_o[e_o f_q]$.

(*a*) *For all* $\mu \in \mathcal{N}_o$, $e^*_o \mu | N^*_o = 0$

Let $u \in N^*_o$, $\gamma \in \hat{M}^o|_u$. If γ is tangential to N^*_o, $e_{o*}\gamma = 0$, since $e_o(N^*_o) = \{o\}$. Suppose, therefore, that γ is not tangential to N^*_o, and define $\lambda: R^2 \to T^*_o$, $(\xi, \eta) \to \xi(u + \eta\gamma')$, where $\gamma' \in T^*_o$ is the isomorph of γ. We use temporarily the symbols X, Y, x, y to denote vectors and maps given by

$$X = \partial/\partial\xi, \; Y = \partial/\partial\eta, \; x, y: R^2 \to R, \; (\xi, \eta) \to \xi, \eta, \text{ respectively.} \tag{5·3}$$

Thus, $dx(X) = X(x) = 1$, etc. Now, $\Lambda \overset{\text{def}}{=} e_o \lambda$ maps $R \times \{\eta\}$ for each η onto a geodesic of M. Hence, for any $\mu' \in \mathcal{N}_o$,

$$(\Lambda^*\mu')(X) \equiv \mu_1 \equiv 0, \quad \text{where} \quad \Lambda^*\mu' = \mu_1 \, dx + \mu_2 \, dy, \mu_1, \mu_2 \in \mathscr{F} R^2.$$

On the other hand, $\Lambda_* X|_{(\xi, 0)} = 0$, $\lambda_* X$ then being tangential to N^*_o. Accordingly, on $R \times \{0\}$, we have, since $[X, Y] \equiv 0$,

$$(\Lambda^* \circ d\mu')(X, Y) = 0 = X(\Lambda^*\mu')(Y) - Y(\Lambda^*\mu')(X) = \partial\mu_2/\partial\xi.$$

Thus, $\mu_2 | R \times \{0\}$ is constant. When $\xi = \eta = 0$, $\lambda_* Y = 0$, whence

$$\mu_2 = (\Lambda^*\mu)(Y) = (e^*_o \mu)(\lambda_* Y) = 0.$$

Hence, $\mu_2 = 0$ when $\eta = 0$; in particular, at $(1, 0)$, $\lambda_* Y = \gamma$, so that

$$\mu_2|_{(1,0)} = \mu(e_{o*}\gamma) = (e^*_o \mu)(\gamma) = 0.$$

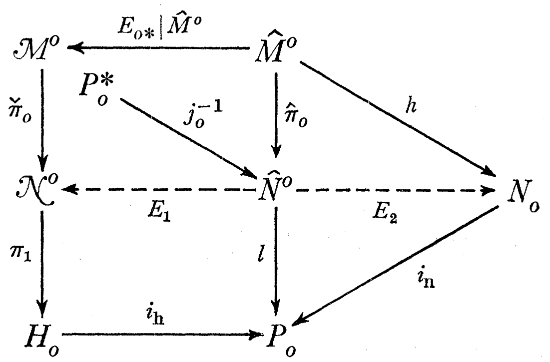
But $\gamma \in \hat{M}^o$ is arbitrary, and so $e^*_o \mu | N^*_o$ is zero. Taking $\gamma = \dot{f}_q(0)$, this gives $e_{o*} \dot{f}_q(0) \in H_o$.

(*b*) $\hat{e}_o\colon P_o^* \to P_o,\ q \to \phi_o[e_o f_q]$, *is well defined*

Let $\pi_{\mathrm{h}}, \pi_{\mathrm{n}}\colon P_o \to H_o, N_o$, respectively. From definition 5·2, proposition 5·1 and (*a*),

$$\pi_{\mathrm{h}}\,\phi_o[e_o f_q] = e_{o*}\,\dot{f}_q(0) = e_{o*}\,\hat{q}.$$

With E_o as in definition 2·7, we have $e_o = \pi E_o$. Since $E_o|N_o^* = 1$, $E_{o*}|\hat{\mathscr{M}}^o$ leaves vectors tangent to N_o^* fixed, while $\check{\pi}_o$, $\hat{\pi}_o$ map them to zero. Accordingly, E_{o*} induces a map $E_1\colon \hat{N}^o \to \mathscr{N}^o$ by commutativity of the first square in the following diagram:



(5·4)

Hence, $$\pi_{\mathrm{h}}\,\phi_o[e_o f_q] = \pi_1\,\check{\pi}_o\,E_{o*}\,\hat{q} = \pi_1\,E_1\,\hat{\pi}_o\,\hat{q} = \pi_1\,E_1 j_o^{-1}\,q.$$

On the other hand, from definition 5·2, $\pi_{\mathrm{n}}\,\phi_o[e_o f_q]$ is that element of N_o which sends $\mu(\in N_o^*)$ to $\frac{1}{2}\,\mathrm{d}_t\,{}^e\mu\dot{f}_q(t)$. By proposition 5·1 and notation 5·2, for any $\psi \in {}'\mu$,

$$\mathrm{d}_t\,{}^e\mu\dot{f}_q(t) = \mathrm{d}_t^2\,{}^e\psi f_q(t) = 2\mathscr{H}[{}^e\mu]\,(\hat{q},\hat{q}).$$

The map $$h\colon \hat{M}^o \to N_o,\ \tau \to \mathscr{H}[{}^e ?]\,(\tau,\tau),$$

induces by commutativity a map $E_2\colon \hat{N}^o \to N_o$. This is because ${}^e\psi$ is constant on N_o^*, so that $\mathscr{H}[{}^e\psi]\,(\sigma,\tau)$ vanishes whenever σ or τ is tangential to N_o^*. Thus,

$$\pi_{\mathrm{n}}\,\phi_o[e_o f_q] = h\hat{q} = E_2 j_o^{-1} q.$$

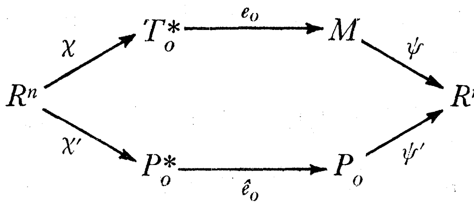
Defining $l = i_{\mathrm{h}}\,\pi_1\,E_1 + i_{\mathrm{n}}\,E_2$, we have

$$\hat{e}_o = l j_o^{-1} = i_{\mathrm{h}}\,\pi_1\,E_1 j_o^{-1} + i_{\mathrm{n}}\,E_2 j_o^{-1}, \tag{5·5}$$

and this proves the result.

The importance of the map $\hat{e}_o$ lies in the facts (1) that its properties are easily computed, using §4 and theorem 6·1 below, (2) that it approximates to e_o near the origin, as will now be shown. First we need

Notation 5·3. Let $\mu^{m+1}, \ldots, \mu^n$ be a basis for N_o^* and choose coordinates $\{x^i\}$ at o such that $x^i(o) = 0$, $x^\lambda \in {}'\mu^\lambda$. Then $\{x^i\}$ will be called *special coordinates at o*. Let $\psi\colon U\{x^i\} \to R^n$ be a special chart at $o \in M$, and assume for convenience that the domain of each x^i is M, i.e. $x^i \in \mathscr{F}$. Set $\eta^i = \mathrm{d}x^i|_o$, define a basis $\{\eta'^i\}$ for P_o^* by $\eta'^\alpha = i^*\eta^\alpha (\in H_o^*)$, $\eta'^\lambda = \eta^\lambda$, where $i\colon H_o \to T_o$ is the inclusion, and let $\{\eta_i'\}$, $\{\eta_i\}$ be the dual bases for P_o, T_o, respectively. Define maps $\epsilon_o \equiv \psi e_o \chi$, $\hat{\epsilon}_o \equiv \psi' \hat{e}_o \chi'$ by



where $\chi(y_1, \ldots, y_n) \equiv y_i \eta^i$, $\chi'(y_1, \ldots, y_n) \equiv y_i \eta'^i$, $\psi'(u^i \eta'_i) \equiv (u^1, \ldots, u^n)$. If $y \in R^n$ we write $\epsilon_o(y) = (\epsilon_o^1(y), \ldots, \epsilon_o^n(y))$, and similarly for $\hat{\epsilon}_o$. Recall that Greek suffixes $\alpha, \beta, \ldots, \epsilon$ range from 1 to m, and $\kappa, \ldots, \rho$ from $m+1$ to n.

Proposition 5·3. With notation 5·3, we have on any open ball $\{(y_1, \ldots, y_n) \in R^n | y_1^2 + \ldots + y_n^2 < \delta^2, \delta > 0\}$,

$$\epsilon_o^\alpha(y_1, \ldots, y_n) = \hat{\epsilon}_o^\alpha(y_1, \ldots, y_n) + O(|y|^2),$$
$$\epsilon_o^\lambda(y_1, \ldots, y_n) = \hat{\epsilon}_o^\lambda(y_1, \ldots, y_n) + O(|y|^3),$$

where $|y| \equiv \left(\sum_1^m y_\alpha^2\right)^{\frac{1}{2}}$. The functions $\hat{\epsilon}_o^\alpha$ and $\hat{\epsilon}_o^\lambda$ are the linear and quadratic terms, respectively, in the Taylor formula for ϵ_o^α, ϵ_o^λ in powers of $(y_1, \ldots, y_m)$.

Proof. Clearly, ϵ_o is C^∞, and if $\mathscr{R}_k^i$ denotes the kth remainder in the ϵ_o^i expansion, we have

$$\epsilon_o^i(y_1, \ldots, y_n) = y_\alpha \frac{\partial \epsilon_o^i}{\partial y_\alpha}\bigg|_{y'} + \tfrac{1}{2} y_\alpha y_\beta \frac{\partial^2 \epsilon_o^i}{\partial y_\alpha \, \partial y_\beta}\bigg|_{y'} + \ldots + \mathscr{R}_k^i, \tag{5·6}$$

where $y' \equiv (0, \ldots, 0, y_{m+1}, \ldots, y_n)$. The term $\epsilon_o^i(y')$ is zero, because $e_o(N_o^*) = \{o\}$. In (5·6), writing $\tilde{y} = y_\alpha \, \partial/\partial y_\alpha$, $\tilde{\eta}^i = \chi_*(\partial/\partial y_i)$ $(\in \hat{M}^0|_{\chi y'})$,

$$\tilde{y} \circ \epsilon_o^i|_{y'} = \tilde{y}(x^i e_o \chi) = \mathrm{d}x^i(e_{o*} \chi_* \tilde{y}) = \mathrm{d}x^i(e_{o*}(y_\alpha \tilde{\eta}^\alpha))$$
$$= \eta'^i(\pi_\mathrm{h} \hat{e}_o j_o \hat{\pi}_o(y_\alpha \tilde{\eta}^\alpha)) = \eta'^i(\pi_\mathrm{h} \hat{e}_o(y_j \eta'^j)) = \eta'^i(\pi_\mathrm{h} \hat{e}_o \chi'(y_1, \ldots, y_n)).$$

This vanishes if $i = m+1, \ldots, n$. On the other hand,

$$\tilde{y} \circ \epsilon_o^\alpha|_{y'} = \eta'^\alpha(\hat{e}_o \chi'(y_1, \ldots, y_n)) = \hat{\epsilon}_o^\alpha(y_1, \ldots, y_n),$$

giving one of the equations required. By (5·4) and (5·5), $\pi_\mathrm{h} \hat{e}_o j_o \hat{\pi}_o = E_2 \hat{\pi}_o = h$, and because $\tilde{y} \circ \epsilon_o^\lambda|_{y'} = 0$, we have

$$y_\alpha y_\beta \frac{\partial^2 \epsilon_o^\lambda}{\partial y_\alpha \, \partial y_\beta}\bigg|_{y'} = \tilde{y} \circ \tilde{y} \circ (x^\lambda e_o \chi) = \tilde{y} \circ \tilde{y} \circ \{{}^e x^\lambda (y_j \eta^j)\}$$
$$= 2\mathscr{H}[{}^e x^\lambda](\chi_* \tilde{y}, \chi_* \tilde{y}) = 2\eta'^\lambda(\pi_\mathrm{h} \hat{e}_o j_o \hat{\pi}_o(y_\beta \tilde{\eta}^\beta))$$
$$= 2\eta'^\lambda(\hat{e}_o(y_j \eta'^j)) = 2\hat{\epsilon}_o^\lambda(y_1, \ldots, y_n),$$

as required.

Corollary 5·1. $\hat{\epsilon}_o$, and hence $\hat{e}_o$, is C^∞.

If $\mathscr{J}$, $\hat{\mathscr{J}}$ are corresponding principal minors of the Jacobians $\partial \epsilon_o^i / \partial y_j$, $\partial \hat{\epsilon}_o^i / \partial y_j$, respectively, containing exactly $r\,(=0, 1, \ldots, n-m)$ of the last $n-m$ rows, then $\hat{\mathscr{J}}$ is a homogeneous polynomial of degree $2r$ in $(y_1, \ldots, y_m)$. Consequently, defining

$$\mu = z_\lambda \eta^\lambda \in N_o^*, \; \tilde{z} = z_\alpha \, \partial/\partial y_\alpha + \zeta_\lambda \, \partial/\partial y_\lambda, \; z_1, \ldots, z_n, \; \zeta_{m+1}, \ldots, \zeta_n \in R,$$

we have $$(\tilde{z})^{2r} \circ \mathscr{J}|_{\chi^{-1}(\mu)} = (\tilde{z})^{2r} \circ \hat{\mathscr{J}}|_{\chi^{-1}(\mu)} = (2r)!\, \hat{\mathscr{J}}(z_1, \ldots, z_n),$$

the result being independent of the ζ's. Let τ be a non-null constant vector field on T_o^*. Choosing $\tilde{z} = (\chi^{-1})_* \tau$, we have

$$\tau^{2r} \circ (\mathscr{J} \circ \chi^{-1})|_\mu = (2r)!\, \hat{\mathscr{J}}(z_1, \ldots, z_n). \tag{5·7}$$

Observe that $$\partial \epsilon_o^i / \partial y_j|_{\chi^{-1}(\mu)} = (\partial/\partial y_j)\{x^i e_o(y_k \eta^k)\}|_{\chi^{-1}(\mu)} = \tilde{\eta}^j(x^i e_o)|_\mu. \tag{5·8}$$

Corollary 5·2. Assume that, for some $\epsilon > 0$, proposition 4·2*a* is true for the matrix-valued function $(\hat{J}^{ij}) \equiv (\partial \hat{e}^i_o(y_1, \ldots, y_n)/\partial y_j)$. Then the first non-vanishing derivative (which may be the 0th) in every non-null direction τ of every principal minor of the matrix function $\mu \to (\tilde{\eta}^j \circ (x^i e_o))|_\mu$ is positive for all $\mu \in N^*_o(\epsilon)$.

The following result will be needed in §7.

Proposition 5·4. The map $\hat{e}$: $P^* \to P$, defined in proposition 5·2, is C^∞.

Proof. Let π_h, π_n, π'' project P onto H, N, M, respectively. Let π': $P^* \to M$, and let i_h, i_n denote the inclusions H, $N \to P$. Since $j\hat{\pi}$: $\hat{M} \to P^*$ is C^∞ ,of maximal rank and onto, a function r: $P^* \to R$ is C^∞ iff $rj\hat{\pi}$ is C^∞. The proposition therefore holds if $\psi \in \mathscr{F}P$ implies $\psi\hat{e}j\hat{\pi} \in \mathscr{F}\hat{M}$. In case $\psi = \sigma\pi''$, $\sigma \in \mathscr{F}M$, we have $\sigma\pi''\hat{e} = \sigma\pi' \in \mathscr{F}P^*$, since $\hat{e}$ respects fibres. On the other hand, the set of C^∞ functions hom $(P, R) \cup \pi''^*\mathscr{F}M$ contains a set of charts covering P. Thus, we can assume ψ to be a homomorphism, so that $\psi = \psi i_h \pi_h + \psi i_n \pi_n$. By (*b*) of the proof of proposition 5·2,

$$\pi_h \hat{e} j\hat{\pi} = e_* | \hat{M}: \hat{M} \to H.$$

Since the inclusions $\hat{M} \to TT^*$ and i_h are C^∞, $\psi i_h \pi_h \hat{e} j\hat{\pi}$ is C^∞.

Finally, a homomorphism ψi_n: $N \to R$ is an element $\mu \in \mathscr{N}$. To prove that $\psi i_n \pi_n \hat{e}$ is C^∞, let $\tau \in \hat{M}$; we have (cf. proof (*a*) above)

$$\psi i_n \pi_n \hat{e} j\hat{\pi}(\tau) = \mathscr{H}[{}^e\mu]\,(\tau, \tau).$$

Introduce a C^∞ Riemannian metric on T^* and let $\Phi \in \mathscr{F}^1 TT^*$ be its geodesic spray. The 1-form ${}^e\mu$ on T is also a C^∞ scalar on TT^*, so the function λ: $TT^* \to R$, $y \to \frac{1}{2}\Phi \circ {}^e\mu|_y$ is C^∞. To interpret $\Phi \circ {}^e\mu|_y$, choose an embedding $\hat{y}$ of y, namely a C^∞ section $\hat{y}$: $T^* \to TT^*$ such that $y \in \operatorname{im}\hat{y}$, $\hat{y}_* y = \Phi|_y$. Then, since $\hat{\pi}_*(\Phi|_y) = y$, where $\hat{\pi}$: $TT^* \to T^*$,

$$\Phi \circ {}^e\mu|_y = (\hat{y}_* \hat{\pi}_* \Phi) \circ {}^e\mu|_y = y \circ (\hat{y}^* \, {}^e\mu) = y({}^e\mu\hat{y}) = 2\mathscr{H}[{}^e\mu]\,(y, y)$$

if $y \in \hat{M}$, by proposition 5·1 and notation 5·3. Accordingly, $\lambda|\hat{M} = \psi i_n \pi_n \hat{e} j\hat{\pi}$, whence the latter is C^∞.

6. Proof of the identity $\hat{e} = e'$

Our object here is to prove theorem 6·1 which, by proposition 5·3, can be interpreted by saying that the map e_o behaves like the map e'_o of §4 near N^*_o. For this purpose, we need several lemmas. Throughout this section $o \in M$ will be fixed.

Lemma 6·1. The scalar A of definition 2·1, vanishing on N^*_o, has a positive, semi-definite Hessian $\mathscr{H}[A]$ on N^*_o. $\mathscr{H}[A]$ induces a quadratic function $\mathscr{M}^o \to R$ which projects by $\check{\pi}_o$ onto a quadratic function $\tilde{A}_1$: $\mathscr{N}^o \to R$. Moreover, the 2-form $\mathrm{d}\omega$ on T^* induces a bilinear form $\mathrm{d}\tilde{\omega}_1$ on each fibre of $\mathscr{N}^o$ such that $\mathrm{d}\tilde{\omega}_1(\check{\pi}_o \tau_1, \check{\pi}_o \tau_2) = \mathrm{d}\omega(\tau_1, \tau_2)$ for all τ_1, $\tau_2 \in \mathscr{M}^o|_\mu$ and all $\mu \in N^*_o$.

Proof. The first part is immediate. The last part follows from the fact that, if α_1, α_2, ν_1, $\nu_2 \in \mathscr{M}^o|_\mu$, $\mu \in N^*_o$, where ν_1, ν_2 are tangential to N^*_o, we have

$$\mathrm{d}\omega(\alpha_1 + \nu_1, \alpha_2 + \nu_2) = \mathrm{d}\omega(\alpha_1, \alpha_2),$$

since, for example, by proposition 2·1, if $\hat{\nu}_1 (\in N^*_o)$ is the isomorph of the vertical vector ν_1, $(\mathrm{i}[\nu_1]\,\mathrm{d}\omega)\,(\alpha_2) = \hat{\nu}_1(\alpha_2) = 0$.

LEMMA 6·2. There is a diffeomorphism

$$k:\mathcal{N}_o^o \to T^*H_o \oplus N_o^* = H_o \oplus H_o^* \oplus N_o^* \tag{6·1}$$

determined by the structure $\mathscr{P}(M, H, a)$ with the properties that, if A_1', ω_1' are the forms A', ω' of proposition 3·3 for the case where $B = H_o$, $\Sigma = N_o$, δ given by (4·1) then

(i) $\tilde{A}_1 = k^*A_1'$,

(ii) for any $u \in \mathcal{N}_o^o$ (recall that $\mathcal{N}_o^o$ is a vector bundle over N_o^*) and vertical vectors $\sigma, \tau \in T_u\mathcal{N}_o^o$, we have $\mathrm{d}\omega_1'(k_*\sigma, k_*\tau) = \mathrm{d}\tilde{\omega}_1(\mathrm{t}\sigma, \mathrm{t}\tau)$, where t is the operation which takes a vertical tangent vector in $\mathcal{N}_o^o$ to its isomorph.

Proof. We have projections

$$\pi_1:\mathcal{N}_o^o \to H_o, \quad \pi_3:\mathcal{N}_o^o \to N_o^*. \tag{6·2}$$

For any $\mu_o \in N_o^*$, choose $\mu \in \mathscr{N}_o$ such that $\mu|_o = \mu_o$. Then $\mu\colon U \to N^*(\subset T^*)$ for some open $U(\ni o)$ in M. If μ, $\bar{\mu}$ are two such sections, with $\bar{\mu}|_o = \mu_o$, then, for any $\sigma \in H_o$,

$$\bar{\mu}_*\sigma - \mu_*\sigma \, (\in T_{\mu_o}N^*)$$

is vertical in N^*, viz. tangential to N_o^*. Hence, μ_o induces a homomorphism $s_{\mu_o}\colon H_o \to \mathcal{N}_o^o|_{\mu_o}$ independent of the embedding of μ_o in $\mathscr{N}_o$. Moreover, s_{μ_o} is mono, since $\pi_1 s_{\mu_o} = 1$. Next, with μ, $\bar{\mu}$, σ as above, $\mu - \bar{\mu} = c^\lambda \mu_{(\lambda)}$ for some $c^\lambda \in \mathscr{F}$, $\mu_{(\lambda)} \in \mathscr{N}_o$, $\lambda = 1, 2, \ldots, n-m$, where $c^\lambda|_o = 0$. Then $\mathrm{i}[\sigma]\,\mathrm{d}(\mu - \bar{\mu}) \in N_o^*$, as one easily checks, and so we have a homomorphism $H_o \to H_o^*$ given by $\sigma \to \mathrm{i}[\sigma]\,\mathrm{d}\mu_o + N_o^*$, where $\mathrm{d}\mu_o = \mathrm{d}\mu|_o$ for any $\mu \in \mathscr{N}_o$ such that $\mu|_o = \mu_o$. For any $\nu \in \mathcal{N}_o^o$, let $\mu_o = \pi_3\nu$; then $\pi_1(\nu - s_{\mu_o}\pi_1\nu) = 0$, so that $\nu - s_{\mu_o}\pi_1\nu \in \hat{N}^o|_{\mu_o}$. With p_o as in notation 5·2, we define (using the isomorph $H_o^* \approx M_o^*/N_o^*$)

$$\left.\begin{aligned} &\pi_2:\mathcal{N}_o^o \to H_o^*,\ \nu \to p_o(\nu - s_{\mu_o}\pi_1\nu) + \tfrac{1}{2}\mathrm{i}[\pi_1\nu]\,\mathrm{d}\mu_o + N_o^*, \\ &k:\mathcal{N}_o^o \to T^*H_o \oplus N_o^*,\ \alpha \to \pi_1\alpha + \pi_2\alpha + \pi_3\alpha. \end{aligned}\right\} \tag{6·3}$$

It is immediate that k is a bijection. The proof that k is a diffeomorphism will be omitted.

The projection π_2 can be realized in another way as follows. For any $\mu_o \in N_o^*$, choose $\phi \in {}'\mu_o$ (definition 5·3); then $\mathrm{d}\phi\colon U \to T^*$, some open $U \ni o$, and $(\mathrm{d}\phi)_*|H_o\colon H_o \to T_{\mu_o}T^*$ induces a homomorphism $s'_{\mu_o}\colon H_o \to \mathcal{N}_o^o|_{\mu_o}$ such that $\pi_2 = p_o(1 - s'_{\mu_o}\pi_1)$, as will now be shown. Choose $\sigma, \tau \in H_o$; then $\sigma(\phi) = \tau(\phi) = 0$, and so we can extend σ, τ throughout U in such a way that $\sigma(\phi) \equiv \tau(\phi) \equiv 0$ and $[\sigma, \tau] \equiv 0$ in U (e.g. by taking a chart $\{x^i\}$ where $x^1 = \phi$, and σ, τ the vector fields defined by $\partial/\partial x^2$, $\partial/\partial x^3$). Because $\mathscr{L}[\sigma]\,(\mathrm{d}\phi) \equiv \mathscr{L}[\tau]\,(\mathrm{d}\phi) \equiv 0$, the pairs σ, σ' and τ, τ' are $\mathrm{d}\phi$-related, where σ', τ' are the first-order prolongations of σ, τ in T^*. Choose $\mu \in \mathscr{N}_o$, σ'', $\tau'' \in \mathscr{F}^1T^*$ such that $\mu|_o = \mu_o$, $\sigma''|_{\mu_o} = \mu_*\sigma$, $\tau''|_{\mu_o} = \mu_*\tau$ and also such that σ'', σ and τ'', τ are π-related. Note that σ', σ and τ', τ are π-related. Consider the expression (evaluated at μ_o)

$$\begin{aligned} \mathrm{d}\omega(\mu_*\sigma - (\mathrm{d}\phi)_*\sigma,\ (\mathrm{d}\phi)_*\tau) &= \mathrm{d}\omega(\mu_*\sigma,\ (\mathrm{d}\phi)_*\tau) - ((\mathrm{d}\phi)^*\circ\mathrm{d}\omega)\,(\sigma, \tau) \\ &= \sigma''\omega(\tau') - (\mathscr{L}[\tau']\,\omega)\,(\sigma'') - (\mathrm{d}^2\phi)\,(\sigma, \tau), \end{aligned}$$

using proposition 2·1. Here, $\mathscr{L}[\tau']\,(\omega) \equiv 0$ because τ' is a first-order prolongation (proposition 2·1), $\mathrm{d}^2\phi \equiv 0$ and $\omega(\tau')|_y = y(\pi_*\tau') = y(\pi_*\tau'') = \omega(\tau'')$. Hence,

$$\mathrm{d}\omega(\mu_*\sigma - (\mathrm{d}\phi)_*\sigma,\quad (\mathrm{d}\phi)_*\tau) = \sigma''\omega(\tau'') = (\mu_*\sigma)\,\omega(\mu_*\tau) = \sigma\mu(\tau). \tag{6·4}$$

For any $\alpha, \beta \in R$ there is an orbit $f_{\alpha\beta}$ of the field $\alpha\sigma + \beta\tau$ for which $f_{\alpha\beta}(0) = o$. Accordingly, by proposition 5·1, since $\sigma\phi \equiv \tau\phi \equiv 0$ and $\dot{f}_{\alpha\beta}(0) \in H_o$,

$$(\alpha\sigma+\beta\tau)^2\,\phi|_o = (\alpha\sigma+\beta\tau)\,\mu(\alpha\sigma+\beta\tau)|_o = 0.$$

From the cases where $(\alpha, \beta) = (1,0), (0,1), (1,1)$, we get $\sigma\mu(\tau) + \tau\mu(\sigma) = 0$, so that, $[\sigma, \tau]$ being zero, (6·4) gives

$$\mathrm{d}\omega(\mu_*\,\sigma - (\mathrm{d}\phi)_*\,\sigma,\ (\mathrm{d}\phi)_*\,\tau) = \tfrac{1}{2}\,\mathrm{d}\mu(\sigma, \tau). \tag{6·5}$$

Now, $\qquad \mathrm{d}\mu(\sigma,\tau) = \{\pi^*(\mathrm{i}[\sigma]\,\mathrm{d}\mu)\}\,((\mathrm{d}\phi)_*\,\tau) = \mathrm{d}\omega(\Omega\pi^*\,\mathrm{i}[\sigma]\,\mathrm{d}\mu,\ (\mathrm{d}\phi)_*\,\tau),$

where Ω is as in proposition 2·1. Since $\mu_*\,\sigma - (\mathrm{d}\phi)_*\,\sigma$, $\Omega\pi^*\,\mathrm{i}[\sigma]\,\mathrm{d}\mu$ are vertical vectors, while $\tau|_o$ is arbitrary in H_o, (6·5) implies

$$(\mathrm{d}\phi)_*\,\sigma|_o = \mu_*\,\sigma - \tfrac{1}{2}\Omega\pi^*\,\mathrm{i}[\sigma]\,\mathrm{d}\mu + \psi, \tag{6·6}$$

where ψ is tangential to N_o^*. We conclude that, for each $\sigma \in H_o$,

$$s'_{\mu_o}(\sigma) = s_{\mu_o}(\sigma) - \tfrac{1}{2}\Omega\pi^*(\mathrm{i}[\sigma]\,\mathrm{d}\mu_o + N_o^*),$$

giving $\pi_2 = p_o(1 - s'_{\mu_o}\pi_1)$ as required.

To prove (i), given $\nu \in \mathcal{N}_o^o|_{\mu_o}$, choose $\check{\nu}$, $\tau \in \mathcal{M}^o$ such that $\check{\pi}_o\check{\nu} = \nu$, $\check{\pi}_o\tau = s_{\mu_o}\pi_1\nu$, where $\mu \in \mathcal{N}_o$ is an embedding of μ_o. Then τ is tangential to N^*, whence $\mathscr{H}[A]\,(\tau, \sigma) = 0$ for all $\sigma \in T^*_{\mu_o}T^*$. Accordingly,

$$\tilde{A}_1(\nu) = \mathscr{H}[A]\,(\check{\nu}, \check{\nu}) = \mathscr{H}[A]\,(\check{\nu} - \tau, \check{\nu} - \tau) = A(\zeta),$$

where ζ is the isomorph of the vertical vector $\check{\nu} - \tau$—the last equation follows from the quadratic structure of A. If $i\colon H_o \subset T_o$, we have $i^*\zeta = p_o\check{\pi}_o(\check{\nu} - \tau) = p_o(\nu - s_{\mu_o}\pi_1\nu)$, so that

$$A(\zeta) = \|i^*\zeta\|^2 = \|p_o(\nu - s_{\mu_o}\pi_1\nu)\|^2.$$

On the other hand, by (4·1), (6·3) and proposition 3·1,

$$\begin{aligned} A'_1(k\nu) &= A'_1(\pi_1\nu + \pi_2\nu + \pi_3\nu) \\ &= \|\pi_2\nu + \delta(\pi_3\nu)\|^2 \text{ (evaluated at } \pi_1\nu) \\ &= \|p_o(\nu - s_{\mu_o}\pi_1\nu)\|^2, \end{aligned}$$

the entities $\mathrm{t}_\mathrm{h}^*\,\mathrm{i}[\pi_1\nu]\,\mathrm{d}\mu_o, \mathrm{i}[\pi_1\nu]\,\mathrm{d}\mu_o + N_o^*$ in (4·1), (6·3) being identical up to an obvious isomorphism. Combining the last three results yields (i).

To prove (ii), observe that $k_*\,\sigma$, $k_*\,\tau$ are tangential to T^*H_o at ku, and so, by proposition 2·1, part (v), $\qquad \mathrm{d}\omega'_1(k_*\,\sigma, k_*\,\tau) = (\pi_2\,\mathrm{t}\sigma) \circ (\pi_1\,\mathrm{t}\tau) - (\pi_2\,\mathrm{t}\tau) \circ (\pi_1\,\mathrm{t}\sigma).$

Choose $\check{\sigma}$, $\check{\tau} \in \mathcal{M}^o$ such that $\check{\pi}_o\check{\sigma} = \mathrm{t}\sigma$, $\check{\pi}_o\check{\tau} = \mathrm{t}\tau$, respectively. Define $\check{\sigma}_1$, $\check{\sigma}_2$, $\check{\tau}_1$, $\check{\tau}_2$, such that $\check{\sigma} = \check{\sigma}_1 + \check{\sigma}_2$, $\check{\tau} = \check{\tau}_1 + \check{\tau}_2$ and

$$\check{\sigma}_1 = (\mathrm{d}\phi)_*\,\pi_*\,\check{\sigma}, \quad \check{\tau}_1 = (\mathrm{d}\phi)_*\,\pi_*\,\check{\tau}, \quad \phi \in {}'(\pi_3\,u),$$

noting that $\mathcal{M}^o \subset TT^*$. We have $\mathrm{d}\omega(\check{\sigma}_1, \check{\tau}_1) = \mathrm{d}^2\phi(\pi_*\,\check{\sigma}, \pi_*\,\check{\tau}) = 0$, $\mathrm{d}\omega(\check{\sigma}_2, \check{\tau}_2) = 0$ $(\check{\sigma}_2, \check{\tau}_2$ being vertical), and $\mathrm{d}\omega(\check{\sigma}_2, \check{\tau}_1) = (\pi_2\,\mathrm{t}\sigma) \circ (\pi_1\,\mathrm{t}\tau)$, by proposition 2·1, $\pi_2\,\mathrm{t}\sigma$ being the isomorph (modulo N_o^*) of $\check{\sigma}_2$. Accordingly

$$\mathrm{d}\tilde{\omega}_1(\mathrm{t}\sigma, \mathrm{t}\tau) = \mathrm{d}\omega(\check{\sigma}, \check{\tau}) = (\pi_2\,\mathrm{t}\sigma) \circ (\pi_1\,\mathrm{t}\tau) - (\pi_2\,\mathrm{t}\tau) \circ (\pi_1\,\mathrm{t}\sigma),$$

as required.

DEFINITION 6·1. Let W be a vector space over R. There is a vector field w on W defined, for any $\phi \in \mathscr{F}W$, by

$$w(\phi)|_{\mathscr{W}} = \frac{\mathrm{d}}{\mathrm{d}t}\phi(\mathscr{W}+t\mathscr{W})|_{t=0}.$$

We call w the *natural* tangent field on W. The natural tangent field on a vector bundle will be the vertical tangent field which is natural for each fibre. In particular, for T^*, P^* we denote the natural tangent fields by v, v', respectively. (Evidently, $v = \Omega(\omega)$.) The *ray* through $\mathscr{W} \in W$ will be the map $r_{\mathscr{W}}: R \to W$, $t \to e^t \mathscr{W}$. Rays are orbits of the natural tangent field.

LEMMA 6·3. Let a linear function $\alpha: T^*H_o \oplus N_o^* \to R$ such that $\alpha(N_o^*) = 0$ be given. There exists in M a neighbourhood U of o and a C^∞ function $\beta: \pi^{-1}U \to R$ such that, for each $\lambda \in \mathscr{M}^o$,

$$\alpha\kappa(\lambda) = \lambda(\beta), \tag{6·7}$$

$$\lambda v(\beta) = v_1'(\alpha)|_{\kappa\lambda}, \tag{6·8}$$

where $\kappa = k\check{\pi}_o$ and v_1' is the projection into $T^*H_o \oplus N_o^*$ of the natural field v' on T^*P_o (under the map corresponding to π_3 in (3·4)). If $\alpha(H_o \oplus N_o^*) = 0$, β is a bundle homomorphism (namely a tangent field $U \to T$), and $\beta | T_o^* \in H_o$.

Proof. Choose special coordinates $U\{x^i\}$ at o; they induce coordinates $\pi^{-1}U\{\pi^*x^i, y_j\}$ in T^* such that $y_\nu | N_o^* \in N_o$, $\nu = m+1, \ldots, n$. For each $\mu = \mu_\nu \,\mathrm{d}x^\nu|_o \in N_o^*$ $(\mu_{m+1}, \ldots, \mu_n \in R)$ we have an element $\mu_\nu x^\nu \in {}'\mu$ and a section $\phi_\mu: U \to N^*$, $u \to \mu_\nu \,\mathrm{d}x^\nu|_o$. There is a covector $\hat{\alpha} \in T_0^* T^*$ (0 = zero element of T_o^*) such that $\hat{\alpha}(\lambda) = \alpha\kappa(\lambda)$ for all $\lambda \in \mathscr{M}^o|_0$. Let $\hat{\alpha} = \alpha_i \mathrm{d}(\pi^* x^i) + \alpha^j \mathrm{d}y_j$, $\alpha_i, \alpha^j \in R$, and let the last equation define $\hat{\alpha}$ throughout $\pi^{-1}U$. Choose a section $\theta: N_o^* \to \mathscr{M}^o$ as follows: $\theta(0)$ is arbitrary, but vertical in T^*, and, for any $\mu \in N_o^*$, $\theta(\mu)$ is obtained from $\theta(0)$ by translation in T_o^*. The H_o^*-component, $\pi_2 \check{\pi}_o \theta(\mu)$, of $\kappa\theta(\mu)$ is independent of μ, so the same holds for $\alpha\kappa\theta(\mu)$. On the other hand, by the linearity of the y_j, $\hat{\alpha}\theta(\mu) = \alpha^j . \theta(\mu) \circ y_j$ is independent of μ, whence $\hat{\alpha}\theta = \alpha\kappa\theta$—these maps agreeing at 0. Next, let $\theta': N_o^* \to \mathscr{M}^o$ be any section for which $\pi_* \theta'(\mu)$ is independent of μ and also $\theta'(\mu) = \phi_{\mu *} \pi_* \theta'(\mu)$, for each μ, where $\mu = \eta_\nu \,\mathrm{d}x^\nu|_o$. As in the preceding case, $\alpha\kappa\theta'$ is a constant map, while

$$\hat{\alpha}\theta'(\mu) = \alpha_i . \theta'(\mu) \circ (\pi^* x^i) = \alpha_i . \pi_* \theta'(\mu) \circ x^i,$$

which is independent of μ. Hence, $\hat{\alpha}\theta' = \alpha\kappa\theta'$, and since sections of the form θ, θ' form a basis field for $\mathscr{M}^o$, we have $\hat{\alpha} = \alpha\kappa$. Equation (6·7) follows on observing that $\hat{\alpha} = \mathrm{d}\beta$, where $\beta = \alpha_i \pi^* x^i + \alpha^j y_j$.

To prove (6·8), note that, at any point $u = u_1 + u_2 + \mu$ of $T^*H_o \oplus N_o^*$ $(u_1, u_2, \mu \in H_o, H_o^*, N_o^*$, respectively), v_1' is the tangent vector isomorphic to $u_2 + \mu$, and so, since $\alpha(\mu) = 0$, we have $v_1'(\alpha) = \alpha(u_2)$. Hence, taking $u = \kappa\lambda$, where $\lambda = \lambda_1 + \lambda_2 \in \mathscr{M}^o|_\mu$, $\lambda_1 = \phi_{\mu *} \pi_* \lambda$, we have $\kappa(\lambda_2) = u_2 + \mu$. Accordingly,

$$v_1'(\alpha)|_{\kappa\lambda} = \alpha\kappa(\lambda_2) = \lambda_2(\beta) = \alpha^j \lambda(y_j).$$

On the other hand, $v(\beta) = \alpha^j y_j$, by definition of v, whence

$$\lambda v(\beta) = \alpha^j \lambda(y_j) = v_1'(\alpha)|_{\kappa\lambda}.$$

Lastly, if $\alpha(H_o \oplus N_o^*) = 0$, we have that $\alpha_i = 0$, $i = 1, \ldots, n$. The y_j being linear on each fibre, $\beta = \alpha^j y_j$ is a homomorphism. Moreover, in the third sentence of this proof, λ tangential to N_o^* implies $\alpha\kappa\lambda = 0$. Hence, $\alpha^{m+1} = \ldots = \alpha^n = 0$, so that $\beta = \alpha^\gamma y_\gamma$ and $\beta | T_o^* \in H_o$.

NOTATION 6·1. For any $\eta \in \mathscr{F}_r$, $r = 0, 1, 2, \ldots$, ${}^e\eta$ will denote $e_o^* \eta$.

LEMMA 6·4. Each $\sigma \in P_o^*$ defines canonically a vector in H_o (to be denoted by $\pi_{h*} \hat{e}_{o*} \dot{\sigma}$) as follows. Let $v_o = v | T_o^*$ be the natural tangent field on T_o^*; then

$$(\pi_{h*} \hat{e}_{o*} \dot{\sigma})(\phi) = \hat{\sigma} v_o({}^e\phi), \quad \phi \in \mathscr{F}M,$$

where $\hat{\sigma} (\in \hat{M}^o = TT_o^*)$ is chosen such that $j_o \hat{\pi}_o \hat{\sigma} = \sigma$ (see notation 5·2).

Proof. Since ${}^e\phi$ is constant on N_o^*, $v_o({}^e\phi)$ vanishes on N_o^*, so that $\hat{\sigma} v_o({}^e\phi)$ is independent of the choice of $\hat{\sigma} \in \hat{\pi}_o^{-1} j_o^{-1} \sigma$. Moreover, if $\phi, \psi \in \mathscr{F}M$,

$$\hat{\sigma} v_o({}^e(\phi\psi)) = \hat{\sigma} v_o({}^e\phi {}^e\psi) = {}^e\psi . \hat{\sigma} v_o({}^e\phi) + {}^e\phi . \hat{\sigma} v_o({}^e\psi),$$

as is easily verified. The operator $\pi_{h*} \hat{e}_{o*} \dot{\sigma}$, being linear, is therefore in T_o. By the proof of (*a*) in proposition 5·2, $\phi \in \mu \in N_o^*$ implies that ${}^e\phi$ is stationary on N_o^*. Let $\hat{\sigma}$ be embedded in a C^∞ tangent field on TT_o^*. Then

$$\hat{\sigma} v_o({}^e\phi) = [\hat{\sigma}, v_o]({}^e\phi) + v_o \hat{\sigma}({}^e\phi),$$

where both terms on the right are zero. In particular, $\hat{\sigma}({}^e\phi)$ vanishes on N_o^*, so the second term is zero, v_o being tangent to N_o^*. Accordingly, $\pi_{h*} \hat{e}_{o*} \dot{\sigma} \in H_o$.

LEMMA 6·5. If $f: R \to P_o^*$ is a ray, then, for each $\xi \in R$, the tangent vector at ξ to $\pi_h \hat{e}_o f: R \to H_o$ is just $\pi_{h*} \hat{e}_{o*} \dot{f}(\xi)$, as defined in lemma 6·4.

Proof. Set $q = f(1)$ and choose $\hat{q} \in \hat{M}_o$ such that $j_o \hat{\pi}_o \hat{q} = q$. Writing

$$q = q_1 + q_2, \quad q_1 \in H_o^*, \quad q_2 \in N_o^*,$$

one sees that $f(\xi) = j_o \hat{\pi}_o \hat{f}(\xi)$, where $\hat{f}(\xi)$ is the translate in T_o^* of $e^\xi \hat{q}$ to the point $e^\xi q_2$. For any $\phi \in \mathscr{F}$, the tangent vector to $\pi_h \hat{e}_o f$, evaluated on $d\phi$, is

$$\frac{d}{d\xi}[d\phi(\pi_h \hat{e}_o f(\xi))] = \frac{d}{d\xi}[d\phi(e_{o*} \hat{f}(\xi))] = \frac{d}{d\xi}[\hat{f}(\xi) \circ ({}^e\phi)]. \tag{6·9}$$

Define
$$\zeta: R^2 \to T_o^*, \quad \zeta(\xi, \eta) = e^\xi(q_2 + q_1 \eta),$$

where $\hat{q}_1 \in T_o^*$ is the isomorph of $\hat{q}$. We have, by (6·9), since $\zeta_* \circ \partial/\partial\xi = v_o|_{\zeta(\xi,\eta)}$,

$$\frac{d}{d\xi}[d\phi(\pi_h \hat{e}_o f(\xi))] = \frac{\partial^2}{\partial\xi \, \partial\eta}[\zeta^*({}^e\phi)]|_{\eta=0} = \hat{f}(\xi) \, v_o({}^e\phi).$$

LEMMA 6·6. If $f: R \to T_o^*$ is a ray, $E_o f$ is an orbit of the field $v + \Theta$. Conversely, if $l: R \to T^*$ is a $(v+\Theta)$-orbit such that $\lim_{t\to-\infty} l(t) = 0$, then $l': (0, \infty) \to T^*$, $s \to s^{-1} l(\ln s)$, is a Θ-orbit for which $y = \lim_{s\to 0+} l'(s) \in T_o^*$. Consequently, $E_o y = l(0)$.

Proof. Let $h: R \to T^*$ be the Θ-orbit for which $h(0) = f(1)$; then $\dot{h}(1) = \Theta|_{E_o f(1)}$. By proposition 2·11, $s \to e^t h(e^t s)$ is the Θ-orbit through $e^t h(0) = e^t f(1) = f(t)$, whence $E_o f(t) = e^t h(e^t)$. Writing $z = E_o f(t)$, we have, for any $\phi \in \mathscr{F}T^*$,

$$\{E_{o*} \dot{f}(t)\}(\phi) = \frac{d}{dt}\phi E_o f(t) = \frac{d}{dt}\phi\{e^t h(e^t)\} = v\phi|_z + e^t \frac{\partial}{\partial\sigma}\phi\{e^t h(\sigma)\}|_{\sigma=e^t}.$$

Now,
$$\Theta(\phi)|_z = \frac{\partial}{\partial s}\phi\{e^t h(e^t s)\}|_{s=1} = e^t \frac{\partial}{\partial\sigma}\phi\{e^t h(\sigma)\}|_{\sigma=e^t},$$

which yields the first result.

To prove the second statement, let $\phi \in \mathscr{F}T^*$. Then

$$\dot{l}'(s) \circ \phi = \frac{\mathrm{d}}{\mathrm{d}s}\phi\{s^{-1}l(\ln s)\}$$
$$= -s^{-1}\frac{\partial}{\partial u}\phi\{us^{-1}l(\ln s)\}|_{u=1} + s^{-1}\frac{\partial}{\partial v}\phi\{s^{-1}l(v)\}|_{v\ln s}$$
$$= -s^{-1}v\phi|_{l'(s)} + s^{-1}\frac{\partial}{\partial v}\phi\{s^{-1}l(v)\}|_{v=\ln s}.$$

Hence, by (2·6), since l is a $(v+\Theta)$-orbit, while $\lambda_* v = v$ for all $\lambda > 0$,

$$\frac{\partial}{\partial v}\phi\{s^{-1}l(v)\}|_{v=\ln s} = (v+\Theta)\,(s^{-1*}\phi)|_{l(\ln s)}$$
$$= (s^{-1}{}_* v + s^{-1}{}_*\Theta)\,(\phi)|_{l'(s)}$$
$$= \{v(\phi) + s\,.\,\Theta(\phi)\}|_{l'(s)}.$$

Thus, $\dot{l}'(s) = \Theta|_{l'(s)}$.

Finally, by the usual existence theorem, the Θ-orbit l' can be continued beyond $s = 0$ (in the negative sense), and because $l(\ln s) \to 0$, while $\pi l'(s) = \pi l(\ln s)$, each $s > 0$, we have that $\lim_{s\to 0+} l'(s) \in T^*_o$.

LEMMA 6·7. Let $\{x^i\}$ be special co-ordinates at o, and let $\mu \equiv \mu_i\,\mathrm{d}x^i \in \mathscr{N}_o$, where $\mu_i \in \mathscr{F}$. Then $\mathrm{d}\mu_i \otimes \mathrm{d}x^i|_o$ is a skew-symmetric, bilinear form on H_o.

Proof. It suffices to prove that, for all $\tau \in H_o$, $\tau(\mu_i)\,.\,\tau(x^i) = 0$. Let $[f] \in \tilde{P}$, where $\dot{f}(0) = \tau$; since $\mu_\alpha|_o = 0$, $\alpha = 1, \ldots, m$, we have

$$\mathrm{d}_t\mu\dot{f}(t) = \mathrm{d}_t[\mu_i f(t)\,.\,\dot{f}(t) \circ x^i] = \tau(\mu_i)\,.\,\tau(x^i) + \mu_\nu|_o\,\mathrm{d}_t^2\,x^\nu f(t).$$

Now, $c_\nu x^\nu \in {}'\mu$, where $c_\nu = \mu_\nu|_o$. Hence, by proposition 5·1, $c_\nu\,\mathrm{d}_t^2\,x^\nu f(t) = \mathrm{d}_t\mu\dot{f}(t)$, and the result follows.

NOTATION 6·2. Let X, for the moment, denote one of the spaces P^*_o, T^*_o, T^*, and let ξ denote the following vector fields: the natural tangent fields on P^*_o, T^*_o, and the field $v+\Theta$ on T^*—v_o, v, respectively, denoting the natural tangent fields on T^*_o, T^*. Then ξ generates a group of diffeomorphisms $\chi\colon R \times X \to X$, and we write $\chi = \phi$, ψ, Ψ when $X = P^*_o$, T^*_o, T^*, respectively. The trajectories of ϕ, ψ are rays and, because v_o is E_o-related to $v+\Theta$, we have

$$\Psi_t E_o = E_o \psi_t, \quad \Psi_{t*} E_{o*} = E_{o*}\psi_{t*}, \tag{6·10}$$

where we write $\psi_t(x)$ for $\psi(t, x)$. Recall the definition 2·7 of $E\colon T^* \to T^*$.

LEMMA 6·8. For any $\tau \in \hat{M}^o|_\mu$ $(\mu \neq 0)$ not tangential to N^*_o, the sets $\mathrm{im}\,(t \to \psi_{t*}\tau) \subset \hat{M}^o$ and $\mathrm{im}\,(t \to E_*\psi_{t*}\tau) \subset \mathscr{M}^o$ can be locally embedded in C^∞ vector fields in T^* (the first being vertical) which are E-related.

Proof. Choose a Riemannian metric g on M. Let S be the submanifold of T^* given by $\{y \in T^* | g(y, y) = g(\mu, \mu)\}$, and embed τ in a C^∞ vertical vector field (which we call τ) defined on a neighbourhood U of μ in S. Next, define τ on each T^*_x, $x \in \pi U$, by the rule

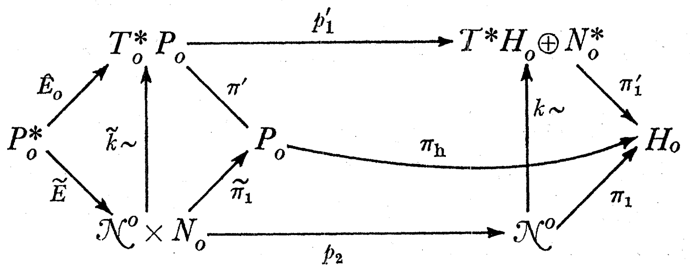
$$\tau|_{\lambda_t(\nu)} = \lambda_{t*}(\tau|_\nu) \quad (\nu \in U),$$

where $\lambda\colon R \times T^* \to T^*$ is the group of diffeomorphisms defined by the field v (so that $\lambda_t | T^*_o = \psi_t$). E (remark 1·1), being a diffeomorphism

$$Q \overset{\mathrm{def}}{\equiv} E_*\tau \quad \text{is } E\text{-related to } \tau.$$

THEOREM 6·1. $\hat{e}_o = e'_o \colon P^*_o \to P_o$.

Proof. With the notation of (5·4), define $\tilde{E}\colon P^*_o \to \mathcal{N}^o_o \times N_o,\ y \to (E_1 j_o^{-1} y,\ E_2 j_o^{-1} y)$. Using (6·1), let $\tilde{k}\colon \mathcal{N}^o_o \times N_0 \to T^*_o P_o$ take (ν_1, ν_2) to $k\nu_1 + \nu_2$, regarding $T^*_o P_o$ for the moment as the vector space $(T^*H_o \oplus N^*_o) \oplus N_o$. Then $\hat{E}_o$, defined by the commutativity of the first triangle in the following diagram, is such that $\hat{e}_o = \pi'\hat{E}_o$. Here, $\tilde{\pi}_1$ takes $(\nu_1, \nu_2) \in \mathcal{N}^o_o \times N_o$ to



(6·11)

$\pi_1\nu_1 + \nu_2$. The rest of the diagram is commutative by construction, p'_1 being the map π_3 of (3·4) for the case $B = H_o$, $\Sigma_o = N_o$. Theorem 6·1 is equivalent to the assertion $\pi'\hat{E}_o = \pi' E'_o$, E'_o being the map E_o for the structure $\mathscr{P}(P_o, H', a'_o)$.

The natural tangent field v' on T^*P_o is obviously p'_1-related to v'_1. Since v' is vertical, it is horizontal with respect to the connexion $\hat{\Gamma}$ defined in the paragraph following proposition 3·2. Hence, by proposition 3·3, $\Theta' + v'$ is the horizontal lift of the field $\Theta'_1 + v'_1$. To prove the theorem we show that, for any ray $f\colon R \to P^*_o$, (i) $\hat{E}_o f$ is horizontal relative to $\hat{\Gamma}$, (ii) $p'_1 \hat{E}_o f$ is a $(\Theta'_1 + v'_1)$-orbit, (iii) $\lim_{t\to-\infty} \hat{E}_o f(t) = 0$ (the origin of T^*P_o), (iv) $\lim_{s\to 0+} \beta_s^{-1}\hat{E}_o f(\ln s) = f(0)$, where $\beta_\lambda\colon T^*P_o \to T^*P_o$ multiplies covectors in P_o by λ.† Statements (i) and (ii) imply that $\hat{E}_o f$ is a $(\Theta' + v')$-orbit, so by lemma 6·6 and (iii), $\hat{E}_o f(0) = E'_o f(0)$. Since $f(0) \in P^*_o$ is arbitrary, we have $\hat{E}_o = E'_o$, whence the result.

Proof of (i). Since $\hat{\Gamma}$ (on T^*P_o) is defined by the 1-form $\pi'^*\gamma$, $\hat{E}_o f$ is horizontal relative to $\hat{\Gamma}$ if $\pi'\hat{E}_o f = \hat{e}_o f$ is horizontal relative to Γ (of proposition 3·1). We therefore show that $\hat{e}_o f$ is an H'-curve. Let $\hat{f}$, ζ be as in the proof of lemma 6·5, and let $X = \partial/\partial\xi$, $Y = \partial/\partial\eta$. It is easy to construct local coordinates $\{u_i\}$ in T^*_o such that $u_i\zeta(\xi,\eta) = \xi\delta^1_i + \eta\delta^2_i$, and hence a local vector field τ on T^*_o which embeds $\zeta_*(\partial/\partial\eta)$. We then have that X, v_o and Y, τ are ζ-related, while $[v_o, \tau] \equiv 0$.

Choose $\mu \in \mathcal{N}_o$ and set $\mu_o = \mu|_o$. Writing $d_\xi = d/d\xi$, we have

$$\mu_o(\pi_n\, \hat{e}_o f(\xi)) = \mathscr{H}[{}^e\mu]\,(\hat{f}(\xi), \hat{f}(\xi)),$$
$$\mu_o(\pi_{n*}\, \hat{e}_{o*}\dot{f}(\xi)) = d_\xi \mathscr{H}[{}^e\mu]\,(\hat{f}(\xi), \hat{f}(\xi)).$$

Let $\{x^i\}$ be special coordinates at o and let $\mu = \mu_i\,dx^i$, where $\mu_i \in \mathscr{F}$ and $\mu_\alpha|_o = 0$, $\alpha = 1, \ldots, m$. Since ${}^e\mu(v_o) \equiv 0$ (rays in T^*_o being mapped by e_o onto images of geodesics in M) at $\zeta(\xi, 0)$, we have

$$\begin{aligned} 0 = \tau^2\, {}^e\mu(v_o) &= \tau^2\{{}^e\mu_i \,.\, v_o({}^e x^i)\} \\ &= 2\tau({}^e\mu_i)\,.\,\tau v_o({}^e x^i) + \mu_\nu|_o \,.\, v_o \tau^2({}^e x^\nu) \quad (\nu = m+1, \ldots, n). \end{aligned} \tag{6·12}$$

Now, $\mu_\nu|_o \,.\, {}^e x^\nu \in {}'\mu$, so that the second term on the right is just

$$2v_o \mathscr{H}[{}^e\mu]\,(\tau, \tau) = 2\mu_o(\pi_{n*}\, \hat{e}_{o*} \dot{f}(\xi)), \tag{6·13}$$

† By notation 1·1, β_λ should be written λ; however, this would be confusing here, since T^*P_o has a vector space structure.

while, by lemmas 6·4, 6·7, the first term equals

$$2\mathrm{F}(\mu_i)\,.\dot{\mathrm{F}}(x^i)|_o = (\mathrm{d}\mu_i \wedge \mathrm{d}x^i)\,(\mathrm{F}, \dot{\mathrm{F}}) = \mathrm{d}\mu(\mathrm{F}, \dot{\mathrm{F}}). \tag{6·14}$$

Here, $\mathrm{F}, \dot{\mathrm{F}}$, respectively, denote $\pi_{\mathrm{h}}\,\hat{e}_o f(\xi)\ (= e_{o*}\tau)$, $\pi_{\mathrm{h}*}\,\hat{e}_{o*}\dot{f}(\xi)$. Combining (6·12), (6.13), (6·14) gives, for each $\mu_o \in N_o^*$,

$$\mu_o(\pi_{\mathrm{n}*}\,\hat{e}_{o*}\dot{f}(\xi)) + \tfrac{1}{2}\mathrm{d}\mu(\pi_{\mathrm{h}}\,\hat{e}_o f(\xi), \pi_{\mathrm{h}*}\,\hat{e}_{o*}\dot{f}(\xi)) = 0.$$

In the notation of (4·1), (4·2), this says that $\mu_o \circ \gamma'(\hat{e}_{o*}\dot{f}(\xi)) = 0$ for each μ_o, i.e. $\hat{e}_o f$ is an H'-curve.

Proof of (ii). If f is null, $j_o^{-1}f(t) \in \hat{N}_o$ is represented for each t by the zero tangent vector to T_o^* at $\mathrm{e}^t\mu$, for some $\mu \in N_o^*$. Hence, in terms of the decomposition (6·1) of $T^*H_o \oplus N_o^*$, one checks readily that $kE_1 j_o^{-1}f(t) = \mathrm{e}^t\mu$—$t \to \mathrm{e}^t\mu$ being a v_1'-orbit. (Recall that $\Theta'|N_o^* = 0$ whence $\Theta_1'|N_o^* = 0$.) Assume now that f is non-null.

In (5·4), $\hat{\pi}_o$ is onto, and one sees, using notation 6·2, that f is of the form

$$t \to j_o\hat{\pi}_o\psi_{t*}\tau \quad (=\phi_t j_o\hat{\pi}_o\tau),$$

for some $\tau \in \hat{\mathscr{M}}^o$, $\tau \neq 0$. Thus, $\qquad \rho(t) = k\check{\pi}_o E_{o*}\,\psi_{t*}\tau.$

Let us modify notations by writing $\tau|_\mu$ instead of τ, the symbol τ now standing for an embedding of $\tau|_\mu$ in a vertical vector field on T_o^*. By lemma (6·8), we assume that $\tau|T_o^*$ is E_o-related to a tangent field Q defined on a neighbourhood of N_o^* in T^*, and that $\tau|\,T_o^*$ and Q are invariant under the action of ψ_t, Ψ_t, respectively. Let $\alpha \in \mathscr{F}T^*H_o \oplus N_o^*$ be a linear function which vanishes on N_o^*. From (6·10) and lemma 6·3, we have

$$\begin{aligned}\dot{\rho}(t)\,(\alpha) &= \mathrm{d}_t[\alpha\kappa E_{o*}\,\psi_{t*}(\tau|_\mu)] = \mathrm{d}_t[(E_{o*}\,\psi_{t*}\,\tau|_\mu)\,(\beta)]\\ &= \mathrm{d}_t[(\Psi_{t*}\,E_{o*}\,\tau|_\mu)\,(\beta)] = \mathrm{d}_t[Q(\beta)|_{\mathrm{e}^t\mu}] = vQ(\beta)|_{\mathrm{e}^t\mu},\end{aligned} \tag{6·15}$$

where $\kappa = k\check{\pi}_o$, $\mathrm{d}_t = \mathrm{d}/\mathrm{d}t$. By (2·5) and proposition 2·1, $2\Theta = \Omega \circ \mathrm{d}A$, whence (recall $\Gamma \equiv \Theta + v$)

$$2[\Gamma, Q] = 2[v, Q] + [\Omega \circ \mathrm{d}A, Q] = 0.$$

On N_o^*, $\mathrm{d}A = 0$, so that, by the rules for Lie derivation,

$$[Q, \Omega \circ \mathrm{d}A] = \Omega \circ \mathrm{d}(QA) = 2[v, Q].$$

Hence, if $F \in \mathscr{F}^1T^*$ is arbitrary, we have on N_o^*

$$2\,\mathrm{d}\omega(F, [v, Q]) = FQ(A) = 2\mathscr{H}[A]\,(F, Q). \tag{6·16}$$

[The path $t \to \pi_*(Q|_{\mathrm{e}^t\mu})$ in T is a Jacobi field over the null geodesic $t \to o$ (parametrized by e^t) and (6·16) is the differential equation which it satisfies.] By lemmas 6·1 and 6·2 (i), for all $\sigma \in \mathscr{M}^o$,

$$\mathscr{H}[A]\,(\sigma, \sigma) = A_1'(k\check{\pi}_o\sigma) = A_1'(\kappa\sigma).$$

Let $\nu \in T_\sigma\mathscr{M}^o$ be vertical in $\mathscr{M}^o$ so that $\kappa_*\,\nu$ is tangential to T^*H_o. Then, since $\mathscr{H}[A] \in \mathscr{F}\mathscr{M}^o$ is quadratic, the right-hand side of (6·16) can be written

$$\nu \circ \mathscr{H}[A] = 2\mathscr{H}[A]\,(\mathrm{t}\nu, \sigma) = (\kappa_*\,\nu)\,A_1'|_{\kappa\sigma}, \tag{6·17}$$

where $\sigma = Q|_{\mathrm{e}^t\mu}$, $\mathrm{t}\nu = F$.

Now, $[v, Q]|\,|_\mu \in \mathscr{M}^o$. In fact, if $\mu_1 \in \mathscr{N}_o$ is an embedding of μ,

$$-(\pi^*\mu)\,([v, Q]) = \{\mathrm{d}\mu_1^*(v, Q) - v\mu_1^*(Q) + Q\mu_1^*(v)\}|_\mu,$$

where $\mu_1^* = \pi^*\mu_1$. The right-hand side vanishes because v is vertical and because

$$v\mu_1^*(Q)|_\mu = v_o{}^e\mu(\tau),$$

where $v|N_o^* = E_{o*}v_o = v_o$. Accordingly, the left-hand side of (6·16) can, by lemmas 6·1, 6·2 (ii), be expressed as

$$2\,\mathrm{d}\tilde{\omega}_1(\check{\pi}_o\mathrm{F}, \check{\pi}_o[v, Q]) = 2\,\mathrm{d}\omega_1'(\kappa_* \nu, \kappa_* \Upsilon), \tag{6·18}$$

where $\Upsilon \in T_Q\mathscr{M}^o$ is a vertical vector such that $\mathrm{t}\Upsilon = [v, Q]$. With the linear function α of (6·15) we have, since $\kappa_*\Upsilon$ is the isomorph of $\kappa\mathrm{t}\Upsilon$,

$$(\kappa_*\Upsilon)(\alpha) = \alpha\kappa[v, Q] = [v, Q](\beta) = vQ(\beta) - Qv(\beta).$$

By (6·8) and (6·15), therefore,

$$\dot{\rho}(t)\circ\alpha = vQ(\beta) = (\kappa_*\Upsilon)(\alpha) + v_1'(\alpha), \tag{6·19}$$

the right-hand side being evaluated at $\kappa(Q|_{\mathrm{e}^t\mu}) = \rho(t)$. For each t, the component of $v_1'|_{\rho(t)}$ tangential to N_o^* is the isomorph of $\mathrm{e}^t\mu$. But this is also easily seen to be the N_o^* component of $\dot{\rho}(t)$. Hence $\dot{\rho}(t) - v_1'$ is tangential to T^*H_o. By proposition 3·3, $\omega_1' = \pi_5^*\omega_1$, so that $\mathrm{i}[\xi]\,\mathrm{d}\omega_1' = 0$ for all ξ tangential to N_o^*. A possible α in (6·19) is therefore given by $\mathrm{d}\alpha = \mathrm{i}[\kappa_*\nu]\,\mathrm{d}\omega_1'$. With this α, equations (6·16) to (6·19) yield

$$2\,\mathrm{d}\omega_1'(\kappa_*\nu, \kappa_*\Upsilon) = 2\,\mathrm{d}\omega_1'(\kappa_*\nu, \dot{\rho}(t) - v_1') = (\kappa_*\nu)(A_1'|_{\rho(t)}).$$

However, all vectors tangent to T^*H_o are of the form $\kappa_*\nu$, and so, by (3·5) $\dot{\rho}(t) = (v_1' + \Theta_1')|_{\rho(t)}$, as required.

Proof of (iii) *and* (iv). Choose $\hat{q} \in \hat{M}^o|_\mu$ such that $f(0) = j_o\hat{\pi}_o\hat{q}$. Let $\bar{q}$ be the constant tangent field on T_o^* for which $\bar{q}|_\mu = \hat{q}$. With $s = \mathrm{e}^t$, we have $sq = j_o\hat{\pi}_o(s\bar{q}|_{s\mu})$, so, taking h as in (5·4),

$$\begin{aligned}\hat{E}_o(sq) &= \kappa E_{o*}(s\bar{q}) + h(s\bar{q})\\ &= b_s\kappa E_{o*}(\bar{q}) + s^2h(\bar{q}),\end{aligned}$$

where $\bar{q}$ means $\bar{q}|_{s\mu}$ and $b_s\colon T^*P_o \to T^*P_o$ is the operation of multiplying the H_o- and H_o^*-components of a point by s. As $t \to -\infty$, $s \to 0+$ and $\hat{E}_o(sq) \to 0$ (observe that the N_o^* component of $\hat{E}_o(sq)$ is $s\mu$), which proves (iii).

Finally, we have

$$\beta_s^{-1}\hat{E}_o(sq) = b_s'\kappa E_{o*}(\bar{q}|_{s\mu}) + s^2h(\bar{q}|_{s\mu}),$$

where $b_s' = \beta_s^{-1}b_s$ is the operation of multiplying the H_o- and N_o^*-components of a point $y \in T^*P_o$ by s, s^{-1}, respectively. Accordingly, as $s \to 0+$, $\beta_s^{-1}\hat{E}_o(sq) \to \mathscr{Q}$, where $\mathscr{Q}(\in T_o^*P_o)$ is the H_o^*-component of $\kappa E_{o*}(\bar{q}|_o)$, plus μ. The H_o^*-component of $\kappa E_{o*}(\bar{q}|_o)$ is defined by the values of $\alpha\kappa E_{o*}(\bar{q}|_o)$ on the linear functions α which vanish on $H_o \oplus N_o^*$. By lemma 6·3, for such an α, β is a bundle homomorphism for which $\beta|T_o^* \in H_o$. Hence, if $\tilde{q} \in T_o^*$ is the isomorph of $\bar{q}|_o$,

$$\begin{aligned}\alpha\kappa E_{o*}(\bar{q}|_o) = \beta E_{o*}(\bar{q}|_o) &= \lim_{s\to 0} s^{-1}[\beta E_o(s\tilde{q}) - \beta E_o(0)]\\ &= \lim_{s\to 0}\beta[s^{-1}E_o(s\tilde{q})].\end{aligned}$$

By lemma 6·6, $s \to s^{-1}E_o^{-1}(s\tilde{q})$ is a Θ-orbit for which $\lim_{s\to 0} s^{-1}E_o(s\tilde{q}) = \tilde{q}$. Accordingly, $\alpha\kappa E_{o*}(\bar{q}|_o) = \beta(\tilde{q}) = \mathrm{d}\beta(\bar{q}|_o) = \alpha\kappa(\bar{q}|_o) = \alpha(\hat{q})$ (α vanishing on $H_o \oplus N_o^*$) $= \alpha j_o\hat{\pi}_o\hat{q}$ (since $\check{\pi}_o|\hat{M}^o = \hat{\pi}_o, k|\hat{N}^o = j_o$) $= \alpha q$. Since the N_o^*-components of $\mathscr{Q}$ and q are both μ, we have $\mathscr{Q} = q = f(0)$.

7. The conjugate locus near the origin

In this section we investigate the behaviour of the exponential map e_o near an arbitrary point o in an arbitrary parabolic space of maximal co-rank.

In terms of notations 4·3, 5·3, theorem 6·1 asserts that $\epsilon'_o = \hat{\epsilon}_o$. From corollary 4·1 and proposition 5·3, we therefore have (cf. notation 4·2).

Theorem 7·1. There is an open set $\mathscr{B}''_o \subset T^*_o$ such that (i) $\mathscr{B}''_o \supset N^*_{o1}$ (ii) $\mathscr{B}''_o \setminus N^*_o$ contains no conjugate points of e_o.

Thus $e_o|\mathscr{B}''_o \setminus N^*_o$ is a local diffeomorphism. Again, by proposition 4·2a, 5·3 and theorem 6·1, there is an open set $\mathscr{B}_o \subset T^*_o$ such that (a) $\mathscr{B}_o \ni 0$, (b) every principal minor of the Jacobian $(\partial\epsilon^i_o(y)/\partial y_j)$, where $y = (y_1, \ldots, y_n)$, is strictly positive on $\chi^{-1}(\mathscr{B}_o \setminus N^*_o)$, ($c$) $(\partial\epsilon^\alpha_o(y)/\partial y_\beta)$ is positive-definite for all $y \in \chi^{-1}(\mathscr{B}_o \cap N^*_o)$. Without loss, assume $\mathscr{B}_o$ to be convex.

Theorem 7·2. $e_o|\mathscr{B}_o \setminus N^*_o$ is a diffeomorphism.

Proof. By theorem 7·1 we only have to prove it 1—1. Let $z_1, z_2 \in \mathscr{B}_o$, where $z_1 \neq z_2$, $e_o(z_1) = e_o(z_2)$, and set $(u_1, \ldots, u_n) = \chi^{-1}(z_1 - z_2)$, observing that χ is linear. Consider the function $w \equiv u_i \epsilon^i_o \in \mathscr{F}\chi^{-1}(\mathscr{B}_o)$. Since $w\chi^{-1}(z_1) = w\chi^{-1}(z_2)$, we have for some θ on the open segment $(\chi^{-1}z_1, \chi^{-1}z_2)$, by the mean value theorem,

$$0 = u_i \,\partial w(y)/\partial y_i|_{y=\theta} = u_i u_j \,\partial\epsilon^i_o/\partial y_j|_\theta, \quad y = (y_1, \ldots, y_n), \tag{7·1}$$

giving a contradiction unless $\theta \in \chi^{-1}N^*_o$. By proposition 5·3, $\partial\epsilon^i_o/\partial y_j|_\theta = 0$ if $\chi\theta = N^*_o$ and i or $j > m$. Hence, (7·1) reduces to $u_\alpha u_\beta \,\partial\epsilon^\alpha_o/\partial y_\beta|_\theta = 0$, so that each $u_\alpha = 0$. Thus, $\chi\theta$, $z_1 - z_2 \in N^*_o$, whence $z_1, z_2 \in N^*_o$, which contradicts the assumption $e_o(z_1) \neq e_o(z_2)$.

Corollary 7·1. If $y \in \mathscr{B}_o \setminus N^*_o$, the geodesic arc $I \to M$, $t \to e_o(ty)$ minimizes J_2 in the set of H-paths $I \to e_o(\mathscr{B}_o)$ which join o to $e_o(y)$.

This holds because the set of all geodesics $t \to e_o(ty)$, $y \in \mathscr{B}_o \setminus N^*_o$ forms a Mayer field covering $e_o(\mathscr{B}_o \setminus N^*_o)$.

Remark. Cartan's uniqueness proof (1951, p. 358) is false, as is the result stated. Consider the counter-example $R^2 \to R^2$, $z \to w = e^z$, $z = x + y\surd{-1}$. The Jacobian $|dw/dz|^2 = e^{2x} \neq 0$, but e^z is not 1—1.

Our final problem in this section is to get information about the subset of M covered by $e_o(\mathscr{B}_o)$. However, in § 8, we require more than this, namely sets covered by $e_o(\mathscr{B}_o)$ as o varies over a compact neighbourhood in M.

Definition 7·1. For $U \subset M$, set

$$\mathscr{N}^2_U \equiv \{\psi \in \mathscr{F}(M \times M) | \psi\iota_u \in {}'\mu, \text{ some } \mu \in N^*_u \quad \text{and} \quad \psi(u, u) = 0 \text{ for all } u \in U\},$$

where ι_u: $M \to M \times M$, $x \to (x, u)$. Also, if $\mu \in \mathscr{N}_U$, set

$$\mu^2_U \equiv \{\psi \in \mathscr{N}^2_U | \,d\psi|_{(u,u)} \in \mu|_u \text{ for all } u \in U\}.$$

Here, and in the sequel, differentiation in a set $V \times V$, where $V \subset M$, is understood to be relative to the first factor, unless indicated otherwise. More explicitly, if $\tau \in T_{(u,v)}(M \times M)$, $d\psi(\tau) \stackrel{\text{def}}{=} d(\psi\iota_v) \circ (p_* \tau)$, where p projects $M \times M$ to its first factor. Note that, if $\phi \in \mathscr{N}^2_U$, then $\phi \in \mu^2_U$, some $\mu \in \mathscr{N}^2_U$; in fact, $\mu|_u = d\phi|_{(u,u)}$.

PROPOSITION 7·1. If $U\{\xi^i\}$ is a coordinate neighbourhood of M and $\mu \in \mathscr{N}_U$, then $\mu^2_U \neq \varnothing$.

Proof. We have $\psi \in \mu^2_U$, where (cf. proposition 5·1)

$$\psi(u,v) = \mu_i(v).\theta^i - \mu_{ij}(v).\theta^i\theta^j \quad (\theta^i = \zeta^i(u) - \zeta^i(v)).$$

Here,
$$\mu = \mu_i.\mathrm{d}\zeta^i,\ \mathrm{d}\mu_i = 2\mu_{ij}.\mathrm{d}\zeta^j,\ \mu_i,\ \mu_{ij} \in \mathscr{F}.$$

PROPOSITION 7·2. There is an open set $\mathscr{B} \subset T^*$, containing the zero section, such that $e|\pi^{-1}x \cap (\mathscr{B}\setminus N^*)$ is a diffeomorphism for all $x \in M$.

Proof. Choose a coordinate neighbourhood $U''\{\zeta^i\}$ of x, then choose U, U' (open) with $x \in U \subset U' \subset U''$ such that

1. There exists a basis $(\mu^{m+1}, \ldots, \mu^n)$ for $\mathscr{N}_{U''}$.
2. $\mathrm{d}\zeta^1 \wedge \ldots \wedge \mathrm{d}\zeta^m \wedge \mu^{m+1} \wedge \ldots \wedge \mu^n$ vanishes nowhere in U'.
3. For some $\xi^\lambda \in (\mu^\lambda)^2_{U''}$, the functions $\xi^i|U \times U$, where $\xi^\alpha(u,v) \equiv \zeta^\alpha(u) - \zeta^\alpha(v)$, $\alpha = 1, \ldots, m$, are such that $\xi^i\iota_v$ are admissible, special coordinate functions at v for all $v \in U$. (This is possible, since the subset of $U' \times U'$ on which $\mathrm{d}\xi^1 \wedge \ldots \wedge \mathrm{d}\xi^n \neq 0$ contains the diagonal.)

Without loss, assume that $\xi^i \in \mathscr{F}(M \times M)$; we can then define maps $\tilde{e}$, Ψ by

$$T^* \xrightarrow{\tilde{e}} M \times M \xrightarrow{\Psi} R^n$$

$$y \to (ey, \pi y),\ (u,v) \to (\xi^1(u,v), \ldots, \xi^n(u,v)).$$

For any $o \in M$, $\tilde{e}|T^*_o$ is the map $y \to (e_o(y), o)$ and $\xi^i\iota_o = x^i$—the x^i being as in notation 5·3. The forms $\mathrm{d}\zeta^\alpha$, μ^λ ($=\mathrm{d}\xi^i$ on the diagonal) naturally induce vertical vector fields $\tilde{\eta}^i$ on $\pi^{-1}U$, constant on each fibre. Hence, the function $y \to \tilde{\eta}^j(\xi^i\tilde{e})|_y$ is equal to the Jacobian element $\partial e^i_o/\partial y_j$, where $o = \pi y$, calculated at $\chi^{-1}(y)$. If τ is a C^∞, vertical, non-null vector field on $\pi^{-1}U$, constant on each fibre, and if $\mathscr{J}_r$ is a principal minor of $\tilde{\eta}^j(\xi^i\tilde{e})$ containing $r(=0, 1, \ldots, n-m)$ of the last $n-m$ rows, the first non-vanishing derivative (the 2rth) of $\mathscr{J}_r$ in the direction τ, calculated on the zero section over U, is positive, by proposition 4·2a, theorem 6·1, corollary 5·2. Since also $\mathscr{J}_r$ vanishes on $N^* \cap \pi^{-1}U$, there is an open set $W \subset \pi^{-1}U$ such that (a) $W \cap T^*_u$, for each $u \in U$, is the interior of a sphere, centre $0|_u$, relative to some Riemannian metric on M, (b) $\tau^{2r} \circ \mathscr{J}_r > 0$ on W for all principal minors $\mathscr{J}_r$ and all non-null τ. Accordingly, the $\mathscr{J}_r$ are positive-definite on $W\setminus N^*$, and the $\mathscr{J}_0$ are positive-definite on $W \cap N^*$. The result therefore follows from theorem 7·2 and the fact the set of U's cover M.

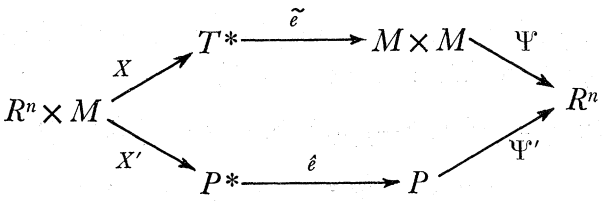
NOTATION 7·1. Let U_0 (open, with compact $\overline{U}_0$) be such that $\overline{U}_0 \subset U$, where U, U'' are sets for which properties 1, 2, 3 (above) hold—so giving C^∞ μ^λ, ξ^i, ζ^j on U. Assume without loss that $(\mathrm{d}\xi^1|_{(v,v)}, \ldots, \mathrm{d}\xi^m|_{(v,v)})$ are orthogonal for each $v \in U$. [In fact, one can replace the (ξ^α) by $(\bar{\xi}^\alpha)$, using the Schmidt orthogonalization process:

$$\bar{\xi}^\alpha(u,v) = \sum_{\beta=1}^{\alpha} c^\alpha_\beta(v).\xi^\beta(u,v),$$

where, for the moment, the symbol $c^\alpha_\beta(v)$ stands for quantities given by

$$c^\alpha_\alpha(v) \sum_{\beta=1}^{\alpha} c^\alpha_\beta(v)\, a^{\beta\gamma}(v) = \delta^{\alpha\gamma}, \quad a^{\beta\gamma}(v) \equiv a(\mathrm{d}\xi^\beta|_{(v,v)}, \mathrm{d}\xi^\gamma|_{(v,v)}).]$$

Define η^i: $U \to T^*$ by $\eta^i|_u = \mathrm{d}\xi^i|_{(u,u)}$, $(\eta_1, \ldots, \eta_n)$ being the dual basis field in T over U, and define η'^i, η'_j in each fibre as in notation 5·3. Define X, X', Ψ' as follows.



(7·2)

$$X(y_1,\ldots,y_n;\,v)=y_i\,\eta^i|_v,\quad X'(y_1,\ldots,y_n;\,v)=y_i\,\eta'^i|_v,\quad \Psi''(u^i\eta'_i)=(u',\ldots,u^n).$$

The diagram is wrong in that the domain of Ψ'' is not P but $\pi''^{-1}U$, $\pi''\colon P\to M$. The maps $\hat{e}$, $\tilde{e}$ are as in propositions 5·2, 7·2, respectively. Next, define

$$\epsilon=\Psi\tilde{e}X|\mathscr{V},\quad \epsilon'=\Psi''\hat{e}X',\quad \mathscr{V}=X^{-1}(\pi^{-1}U\cap e^{-1}U\cap\mathscr{B}),\quad \pi^i\epsilon=\epsilon^i,\quad \pi^i\epsilon'=\epsilon'^i,$$

$\pi^i\colon R^n\to R$ being the ith projection. Obviously, ϵ is C^∞, and by proposition 5·4, so is ϵ'. We define $m_0, m_1, m_2\colon R^n\to R$ by

$$m_1(u_1,\ldots,u_n)=\sum_{\alpha=1}^{m}u_\alpha^2,\quad m_2(u_1,\ldots,u_n)=\sum_{\lambda=m+1}^{n}u_\lambda^2,\quad m_0(u)=\surd[m_1(u)+m_2(u)],$$

and C_ν, Γ_ν $(\nu\geqslant 0)$ by

$$C_\nu=\{u\in R^n|\surd m_2(u)<\nu m_1(u)<\nu^3\}\quad(\nu>0),$$
$$\Gamma_\nu=\{u\in R^n|m_0(u)<\nu\}\quad(\nu>0),\ \Gamma_0=\{(0,\ldots,0)\},$$
$$\Gamma'_\nu=\{u\in R^n|m_0(u)<\nu,\ m_1(u)>0\}\quad(\nu>0).$$

Lemma 7·1. Let $f\colon\overline{\Gamma}_\nu\to R^n$ be a homeomorphism into, for which $f(0)=0$, $0=(0,\ldots,0)$, and $m_0 f(x)\geqslant Km_0(x)$ for all $x\in\overline{\Gamma}_\nu$, where $K>0$. Then $f(\Gamma_\nu)\supset\Gamma_{K\nu}$.

Proof. For any $y\in\partial\Gamma_{K\nu}$, define $t_y=\inf\{t\in R^+|yt\notin f(\Gamma_\nu)\}$. Then $yt_y\in\partial f(\Gamma_\nu)=f(\partial\Gamma_\nu)$, so that, if $x=f^{-1}(yt_y)$, we have $m_0(x)=\nu$. Hence, by hypothesis, $m_0(yt_y)=t_y K\nu\geqslant K\nu$, giving $t_y\geqslant 1$. For any $\eta\in\Gamma_{K\nu}$, we have $\eta=ty$, where $t\in[0,1)$, $y\in\partial\Gamma_{K\nu}$, whence $\eta\in f(\Gamma_\nu)$.

Remark. If f is a diffeomorphism into, one can, by the mean value theorem, take

$$K^2=\inf\sum_{i=1}^{n}\Big[\sum_{j=1}^{n}\zeta_j\,\partial_j(\pi^i f)|_{\xi_i}\Big]^2,$$

where the 'inf' is over $(\xi_1,\ldots,\xi_n,\zeta)\in(\overline{\Gamma}_\nu)^n\times\partial\Gamma_1$, $\zeta=(\zeta_1,\ldots,\zeta_n)$. K will be positive if ν is sufficiently small.

Theorem 7·3. With notation 7·1 there exists $k>0$ such that $\Psi\tilde{e}(\pi^{-1}x\cap(\mathscr{B}\backslash N^*))\supset C_k$ for all $x\in U_0$.

Proof. We first prove that there are numbers k, $l_k>0$ for which

$$\epsilon'(\Gamma'_{l_k}\times\{v\})\supset C_k\quad\text{for all}\quad v\in U_0,\tag{7·3}$$

where $l_k\to 0$ as $k\to 0$. The theorem will then be deduced from proposition 5·3. For each $o\in M$, by (4·13), (4·14), $e'_o|H^*_o=a_o\colon H^*_o\to H_o$, and so, if R^m denotes $\{u\in R^n|m_2(u)=0\}$, $\epsilon'(u,v)=u$ for all $(u,v)\in R^m\times U$. Set $\mathscr{S}^0=\{(u,v)\in R^m\times\overline{U}_0|m_1(u)=1\}$. Since $\mathscr{S}^0$ is compact and $\det\partial\epsilon'^i(u,v)/\partial u_j\neq 0$ for (u,v) near $\mathscr{S}^0$, if k is sufficiently small there is a neighbourhood $\mathscr{S}$ of $\mathscr{S}^0$ in $R^n\times U$ with the properties

(i) $\epsilon'(\mathscr{S}\cap(R^n\times\{v\}))\supset\mathscr{W}_k=\{w\in R^n|\,|m_1(w)-1|<k,m_2(w)<k^6\}$ for all $v\in\overline{U}_0$,

(ii) $0<m_1(u)<2$ and $m_2(u)<\nu_k^2$ for all $(u,v)\in\mathscr{S}$, where $\nu_k\,(>0)$ can be so chosen that $\nu_k\to 0$ as $k\to 0$. [Hint. There is an $\mathscr{S}$ (with $\overline{\mathscr{S}}$ compact) such that, for each v, $\mathscr{S}\cap(R^n\times\{v\})$

is a union of open balls of radius ν_k with centres on $\mathscr{S}^0$. Apply lemma 7·1 and the remark to each ball, using the compactness of $\bar{\mathscr{S}}$.] By the homogeneity of (4·13), (4·14) in y_h of degrees 1, 2, respectively, it follows that, if

$$u = (u_1, \ldots, u_n), \quad u_t = (tu_1, \ldots, tu_m, u_{m+1}, \ldots, u_n), \quad t \in R,$$

then $$m_1\,\epsilon'(u_t, v) = t^2 m_1\,\epsilon'(u, v), \quad m_2\,\epsilon'(u_t, v) = t^4 m_2\,\epsilon'(u, v).$$

Hence, a simple argument using property (i) shows that, if

$$\tilde{\mathscr{S}} = \{(u, v) \in R^n \times U \,|\, \exists\, t > k^{-1} \text{ s.t. } (u_t, v) \in \mathscr{S}\},$$

then $\epsilon'(\tilde{\mathscr{S}} \cap (R^n \times \{v\})) \supset C_k$ for all $v \in \bar{U}_0$. However, by (ii), $\tilde{\mathscr{S}} \subset \Gamma'_{l_k} \times U$, where

$$l_k^2 = 2k^2 + \nu_k^2.$$

Since $l_k \to 0$ as $k \to 0$ the result follows.

For $\delta \in (0, 1)$ sufficiently small, $\Gamma_{4\delta} \times \bar{U} \subset \mathscr{V}$ (notation 7·1), and, with this δ, $\epsilon(u, v) = \epsilon_v(u)$ for all $(u, v) \in \Gamma_{4\delta} \times \bar{U} - \epsilon_v$ being as in notation 5·3. The maps ϵ, ϵ' being C^∞, prop 5·3 implies that

$$m_1(\epsilon(u, v) - \epsilon'(u, v)) < K|u|^4, \quad m_2(\epsilon(u, v) - \epsilon'(u, v)) < K|u|^6, \tag{7·4}$$

for all $(u, v) \in \Gamma_{4\delta} \times \bar{U}_0$ and some $K > 0$, where $|u| = \surd m_1(u)$. Also, setting

$$c^\alpha_\beta = 1, \quad c^\lambda_\alpha = c^\alpha_\lambda = |u|^{-1}, \quad c^\kappa_\lambda = |u|^{-2},$$

we have from proposition 5·3,

$$c^i_j\,\partial\epsilon^i(u_1, \ldots, u_n, v)/\partial u_j = \partial\epsilon'^i(\beta_1, \ldots, \beta_n, v)/\partial\beta_j + O(|u|), \tag{7·5}$$

where $(\beta_1, \ldots, \beta_n)$ denotes $(|u|^{-1}u_1, \ldots, |u|^{-1}u_m, u_{m+1}, \ldots, u_n)$, the O-symbol indicating uniform bounds on $\Gamma_{4\delta} \times \bar{U}_o$. For any $\theta \in [0, 1)$, define

$$W_\theta(u) = \{w \in R^n | m_1(w-u) + m_1(u)\, m_2(w-u) \leqslant \theta m_1(u)\},$$

$$Z(\theta, \delta) = \{(w^1, \ldots, w^n, \zeta_1, \ldots, \zeta_n) \in (W_\theta(u))^n \times R^n | u \in \Gamma'_\delta, \zeta_1^2 + \ldots + \zeta_n^2 = 1\},$$

$$h(\theta, \delta) = \inf \sum_{i=1}^n \Phi^i(w^i, \zeta, v), \quad \Phi^i(u, \zeta, v) \equiv \left[\sum_{j=1}^n c^i_j\,\zeta_j\,\partial\epsilon^i(u_1, \ldots, u_n, v)/\partial u_j\right]^2,$$

where the 'inf' is for all $(w^1, \ldots, w^n, \zeta, v) \in Z(\theta, \delta) \times \bar{U}_0$. By (7·5) and because det $(\partial\epsilon'^i/\partial u_j)$ is non-zero on $\mathscr{S}^0$, a $\delta > 0$ exists for which $h(0, \delta) > 0$. The map $\theta \to h(\theta, \delta)$ being continuous near $\theta = 0$, there is a $\theta \in (0, \delta)$ such that $h(\theta, \delta) > 0$. Let θ now denote this value. We have $0 < \theta < \delta < 1$.

For any $u + \Delta u \in W_\theta(u)$, where $(u, v) \in \Gamma'_\delta \times \bar{U}_0$, we write $z = \epsilon(u, v)$, $z + \Delta z = \epsilon(u + \Delta u, v)$. By the mean value theorem, if $\Delta' u_\alpha = \Delta u_\alpha$, $\Delta' u_\lambda = |u|\,\Delta u_\lambda$,

$$|u|^2 m_1(\Delta z) + m_2(\Delta z) = |u|^2 \sum_{i=1}^n \left[\sum_{j=1}^n c^i_j \left.\frac{\partial\epsilon^i}{\partial u_j}\right|_{\chi_i} \Delta' u_j\right]^2$$
$$\geqslant |u|^2 h(\theta, \delta)\,(m_1(\Delta u) + |u|^2 m_2(\Delta u)),$$

where $\chi_i \in W_\theta(u) \times \{v\}$. This inequality holding for all $u + \Delta u \in W_\theta(u)$, we have from lemma 7·1 that $\epsilon_v(W_\theta(u)) \supset Q_v(u)$, where

$$Q_v(u) = \{z + \Delta z = R^n | z = \epsilon_v(u), \quad |u|^2 m_1(\Delta z) + m_2(\Delta z) \leqslant \theta h(\theta, \delta)\,|u|^4\}.$$

Choose k, l_k in (7·3) such that

$$l_k < \min\left[(\theta h(\theta,\delta)/2K)^{\frac{1}{2}}, \delta\right]. \tag{7·6}$$

We now show that $\epsilon(\Gamma'_{4\delta}\times\{v\}) \supset C_k$ for all $v\in U_0$. Let $w\in C_k$, $v\in U_0$, and define $u(\in\Gamma'_\delta)$ by $\epsilon'(u,v) = w$. If $u+\Delta u\in W_\theta(u)$, we have

$$\begin{aligned}\sqrt{m_1}(u+\Delta u) &\leqslant \sqrt{m_1}(u)+\sqrt{m_1}(\Delta u) \leqslant (1+\theta)\sqrt{m_1}(u) < 2\delta,\\ \sqrt{m_2}(u+\Delta u) &\leqslant \sqrt{m_2}(u)+\theta < 2\delta,\\ \sqrt{m_1}(u+\Delta u) &\geqslant \sqrt{m_1}(u)-\sqrt{m_1}(\Delta u) \geqslant (1-\theta)\sqrt{m_1}(u) > 0,\end{aligned}$$

whence $W_\theta(u) \subset \Gamma'_{4\delta}$. On the other hand, writing $\Delta w = \epsilon'(u,v)-\epsilon(u,v)$, we obtain from (7·5) and (7·6)

$$|u|^2 m_1(\Delta w)+m_2(\Delta w) < 2K|u|^6 < \theta h(\theta,\delta)\,|u|^4,$$

whence $w = \epsilon(u,v)+\Delta w\in Q_v(u) \subset \epsilon(W_\theta(u)\times\{v\}) \subset \epsilon(\Gamma'_{4\delta}\times U_0)$. This completes the proof.

The following result is more easily proved than the last, but it will not be required.

PROPOSITION 7·3. There exists $k' > 0$ such that $\Psi\tilde{\epsilon}(\pi^{-1}x\cap(\mathscr{B}\setminus N^*)) \subset C_{k'}$ for all $x\in U_0$.

8. APPROXIMATION TO HORIZONTAL ARCS BY GEODESIC POLYGONS

DEFINITION 8·1. An H-arc $f\colon I\to M$ will be called C''' if, for all $\phi\in\mathscr{N}^2_{f(I)}$,

$$\lim_{\tau\to t}(\tau-t)^{-3}\phi(f(\tau),f(t))$$

exists, the limit being approached uniformly with respect to $t\in I$.

PROPOSITION 8·1. If f is a C^3 H-arc, f is C'''.

Proof. Follows trivially from definition 8·1 and the mean value theorem.

PROPOSITION 8·2. At each point of M there is a chart $U\{\zeta^i\}$ for which the map ϵ of notation 7·1 has the property

$$\epsilon(y,v) = y \tag{8·1}$$

for all $(y,v)\in(R^m\cap\Gamma_\delta)\times U$, where $\delta > 0$.

Proof. We start with a coordinate neighbourhood $U\{\zeta^i\}$ and functions Ψ, ϵ, etc., as in notation 7·1. For each $v\in U$, by the remark (c) preceding theorem 7·2, $\det\partial\epsilon^\alpha(0,v)/\partial y_\beta \neq 0$. Hence, there are C^∞ maps x_α sending a neighbourhood of the diagonal of $U\times U$ into R, given by

$$\xi^\alpha(u,v) = \epsilon^\alpha(x_1(u,v),\ldots,x_m(u,v),0,\ldots,0,v).$$

Define C^∞ $x_{m+1},\ldots,x_n$ by

$$x_\lambda(u,v) \equiv \xi^\lambda(u,v)-\epsilon^\lambda(x_1(u,v),\ldots,x_m(u,v),0,\ldots,0,v). \tag{8·2}$$

From the second sentence following (7·3),

$$\epsilon'(y,v) = y \quad \text{for all} \quad y\in R^m,\ v\in U, \tag{8·3}$$

whence, by proposition 5·3, $\partial\epsilon^\alpha(0,v)/\partial y_\beta = \delta^{\alpha\beta}$. Hence,

$$\mathrm{d}\xi^\alpha|_{(v,v)} = \sum_\beta \frac{\partial\epsilon^\alpha}{\partial y_\beta}(0,v)\,.\,\mathrm{d}x_\beta|_{(v,v)} = \mathrm{d}x_\alpha|_{(v,v)}, \tag{8·4}$$

and, by (8·2) and (8·4),

$$\mathrm{d}x_\lambda|_{(v,v)} = \mathrm{d}\xi^\lambda|_{(v,v)}-\sum_\beta\frac{\partial\epsilon^\lambda}{\partial y_\beta}(0,v)\,.\,\mathrm{d}x_\beta|_{(v,v)} = \mathrm{d}\xi^\lambda|_{(v,v)}. \tag{8·5}$$

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Thus, $dx_1 \wedge \ldots \wedge dx_n|_{(v,v)} \neq 0$, so U has a cover by neighbourhoods U' for which $U'\{x_i \iota_v\}$ (recall $\iota_v: u \to (u, v)$) is a chart for each $v \in U'$. By (7·4), (8·2) and (8·3)

$$m_2(\Psi(u,v) - X(u,v)) = O(|\Psi(u,v)|^6) \tag{8·6}$$

for all $(u, v) \in U' \times U'$, where $X(u,v) = (x_1(u,v), \ldots, x_n(u,v))$. Since $\xi^\lambda \iota_v \in {}'\mu_v^\lambda$, one deduces from (8·6) and definition 5·3 that $x_\lambda \iota_v \in {}'\mu_v^\lambda$, whence the $x_i \iota_v$ are special coordinates at v.

We now set $\bar{e}^i = x_i \tilde{e} X$ and check that the $\bar{e}^i$ satisfy (8·1) by considering

$$e^\alpha(\bar{e}^1(y,v), \ldots, \bar{e}^m(y,v), 0, \ldots, 0, v), \; y \in R^m.$$

THEOREM 8·1. If $f: I \to M$ is a C''' arc,

(1) there is an $\epsilon > 0$, independent of $t \in I$, such that $f|I \cap (t-\epsilon, t+\epsilon) = e_{f(t)} \tilde{f}_t$, where $\tilde{f}_t$ is an open C^1 arc in $T^*_{f(t)} M$ for which $\tilde{f}_t(t) = 0|_{f(t)}$.

(2) $\dot{\tilde{f}}_t(t)$ is non-null, and, for all $\lambda \in \mathscr{F}T^*$, $(\tau - t)^{-1} (\lambda \tilde{f}_t(\tau) - \lambda \tilde{f}_t(t)) \to \dot{f}_t(t) \circ \lambda$ uniformly in t as $\tau - t \to 0$.

(3) The map $q_f: I \to T^*$, sending t to the isomorph of $\dot{\tilde{f}}_t(t)$ in $T^*_{f(t)}$, is continuous.

(4) $f = \pi g$, where g is a Θ-orbit, implies $g = q_f$.

Proof. Set $F = f \times f: I \times I \to M$. In proving (1), (2) and (3) we assume without loss that the parameter t of f is arc length. For, otherwise, if $s(t)$ denotes arc length, s is C^1 and, by definition 8·1,

$$\lim_{\tau \to t} \{s(\tau) - s(t)\}^{-3} \phi F(\tau, t) = \{s'(t)\}^{-3} \lim_{\tau \to t} (\tau - t)^{-3} \phi F(\tau, t),$$

the limit being uniform in t. Statements (1), (2) and (3) hold similarly on changing from t to s and vice versa. We use notations 7·1, assuming without loss that $f(I) \subset U_0$, where $U\{\xi^i\}$ is a coordinate neighbourhood as in proposition 8·2. By definition 8·1, $(\tau - t)^{-3} \xi^\lambda F(\tau, t)$ is bounded for all $(\tau, t) \in I \times I$, and $\lambda = m+1, \ldots, n$. With d_g as in definition 2·5,

$$|\tau - t|^{-1} d_g F(\tau, t) \to 1$$

uniformly as $t - \tau \to 0$ (use the exponential map $T \to M \times M$ defined by g, diffeomorphic near the zero section). Also,

$$d_g(v_1, v_2)^2 = O(m_1 \Psi(v_1, v_2) + m_2 \Psi(v_1, v_2)) \quad \text{for} \quad (v_1, v_2) \in U_0 \times U_0,$$

so that, from the last two sentences, given δ (sufficiently small, > 0), there exist positive constants k_1, k_2, k_3 such that $|\tau - t| < \delta$ implies

$$[m_2 \Psi F(\tau, t)]^{\frac{1}{3}} < k_1(\tau - t)^2 < k_2\{d_g F(\tau, t)\}^2 < k_3[m_1 \Psi F(\tau, t) + m_2 \Psi F(\tau, t)]. \tag{8·7}$$

Hence, there is a positive ϵ such that $0 < |\tau - t| < \epsilon$ implies $\Psi F(\tau, t) \in C_0$. Statement (1) therefore follows from proposition 7·2 and theorem 7·3, except that we still have to show that (*a*) $\tilde{f}_t(\tau) \to 0$ as $\tau \to t$, (*b*) $\dot{\tilde{f}}_t(t)$ exists, (*c*) $\dot{\tilde{f}}_t(\tau)$ tends to a limit (necessarily $\dot{\tilde{f}}_t(t)$) as $\tau \to t$. We next prove that $s_2 = O(s_1)$ uniformly in t as $\tau \to t$, where s_i means

$$\surd m_i \tilde{\pi} X^{-1} \tilde{f}_t(\tau), \quad i = 1, 2, \tilde{\pi}: R^n \times M \to R^n.$$

For $y \in R^n \times M$, we put $|y| = \surd m_1 \tilde{\pi}(y)$, while, for $y \in T^*$, $|y| = \surd a(y, y)$ as before, so that $|y| = |X^{-1}(y)|$. From definitions 2·5, 8·1 and the second inequality of (8·7),

$$\{d_a F(\tau, t)\}^{-3} \xi^\lambda F(\tau, t) = O(1) \quad \text{for all} \quad (\tau, t) \in I \times I.$$

Now, $|\tilde{f}_t(\tau)| = s_1$ is the length of a geodesic arc joining $f(t)$ to $f(\tau)$, so that $d_a F(\tau,t) \leqslant s_1$. Hence,

$$m_2 \Psi F(\tau,t) = O(s_1^6). \tag{8·8}$$

On the other hand, by proposition 5·3,

$$\left.\begin{aligned} \epsilon^\alpha(y,v) &= b^{\alpha\beta}(Y,v)\,y_\beta + O(|y|^2),\ \epsilon'^\alpha(y,v) = b^{\alpha\beta}(Y,v)\,y_\beta,\\ \epsilon^\lambda(y,v) &= c^{\lambda\beta\gamma}(Y,v)\,y_\beta y_\gamma + O(|y|^3),\ \epsilon'^\lambda(y,v) = c^{\lambda\beta\gamma}(Y,v)\,y_\beta y_\gamma,\end{aligned}\right\} \tag{8·9}$$

where $y = (y_1, \ldots, y_n)$, $Y = (y_{m+1}, \ldots, y_n)$ and the O notation implies uniform bounds on $\mathscr{V}$. From (4·14) and notation 7·1 one computes

$$c^{\lambda\beta\gamma}(Y,v) = c_v^{\lambda\nu\beta\gamma}\,y_\nu + O(m_2(y)), \tag{8·10}$$

where $c_v^{\lambda\nu\beta\gamma} y_\beta y_\gamma$ is a C^∞, non-zero scalar multiple of $a(\Delta\eta'^\lambda|_v \circ y_{\mathrm{h}}, \Delta\eta'^\nu|_v \circ y_{\mathrm{h}})$, $y_{\mathrm{h}} = y_\beta\,\eta'^\beta|_v$. The elements of the matrix $c \equiv (|y|^{-2}\,c_v^{\lambda\nu\beta\gamma} y_\beta y_\gamma)$ and of its inverse (which exists when $|y| \neq 0$) are $O(1)$ on $\mathscr{V}\backslash X^{-1}(N^*)$. Hence, contracting (8·10) with $|y|^{-2} y_\beta y_\gamma$ and operating on it with c^{-1}, using (8·9),

$$y_\lambda = (c^{-1})_{\lambda\nu}\,\epsilon'^\nu(y,v)\,.\,|y|^{-2} + O(m_2(y)),$$

whence, by (8·9), there exist constants K_1, K_2 such that

$$|y_\lambda - (c^{-1})_{\lambda\nu}\,\epsilon^\nu(y,v)\,|y|^{-2}| < K_1|y| + K_2\,m_2(y)$$

for all $(y,v) \in \mathscr{V}\backslash X^{-1}(N^*)$. Hence, by an easy calculation, (A) $(y,v) \in \mathscr{V}$, $m_2\,\epsilon(y,v) = O(|y|^6)$ implies (B) $m_2(y) = O(m_1(y))$. By (8·1) and the identity $e_{f(t)}\tilde{f}_t(\tau) \equiv f(\tau)$ for all $|t-\tau| < \epsilon$, one sees that $X^{-1}\tilde{f}_t(\tau)$ satisfies (A), whence $s_2 = O(s_1)$. It follows that $\tilde{f}_t(\tau) \to 0$ as $\tau \to t$, and that, if $\dot{\tilde{f}}_t(t)$ exists, it is non-null.

We now prove the second statement of (2), that $(\tau-t)^{-1}\,(\lambda\tilde{f}_t(\tau) - \lambda f_t(t))$ tends to a uniform limit as $\tau \to t$. We set.

$$(\sigma_1(\tau,t), \ldots, \sigma_n(\tau,t),\quad f(t)) \equiv X^{-1}\tilde{f}_t(\tau)$$

and observe that, since $\lambda X \in \mathscr{F}(R^n \times U)$, the result is proved if we show that $(\tau-t)^{-1}\,\sigma_i(\tau,t)$ tends to a uniform limit as $\tau \to t$, each i. By (8·1), each term in the Taylor expansion (with remainder) of $\epsilon^\lambda(y_1, \ldots, y_n, v)$ about $y = 0$ contains some y_ν. Hence, by (8·9) and (8·10),

$$\epsilon^\alpha(y,v) = y_\alpha + O(\Sigma y_i^2),\quad \epsilon^\lambda(y,v) = c_v^{\lambda\nu\beta\gamma}\,y_\nu y_\beta y_\gamma + O([\Sigma y_i^2]^2),$$

and so, s_3 denoting $\surd(s_1^2 + s_2^2) = \surd\Sigma\sigma_i^2$, $\sigma_i \equiv \sigma_i(\tau,t)$,

$$\epsilon^\alpha X^{-1}\tilde{f}_t(\tau) = \xi^\alpha F(\tau,t) = \sigma_\alpha + O(s_3^2), \tag{8·11}$$

$$\epsilon^\lambda X^{-1}\tilde{f}_t(\tau) = \xi^\lambda F(\tau,t) = c_{f(t)}^{\lambda\nu\beta\gamma}\,\sigma_\nu\,\sigma_\beta\,\sigma_\gamma + O(s_3^4). \tag{8·12}$$

From corollary 7·1, the geodesic arc $s \to e_{f(t)}(s\tilde{f}_t(\tau))$ of length s_1 which joins $f(t)$ to $f(\tau)$ is shorter than $|\tau - t|$, so that $|\tau-t|^{-1}\,s_1 = O(1)$. Accordingly, since

$$s_2 = O(s_1),\quad |\tau-t|^{-1}\,s_3 = O(1).$$

As $\tau \to t$, $(\tau-t)^{-1}\,\xi^\alpha F(\tau,t)$ tends uniformly to $z^\alpha(t) \equiv \dot{f}(t) \circ (\xi^\alpha \iota_{f(t)})$ and $(\tau-t)^{-3}\,\xi^\lambda F(\tau,t)$ tends to a uniform limit $z^\alpha(t)$, by definition 8·1. Multiplying (8·11) and (8·12) by $(\tau-t)^{-1}$, $(\tau-t)^{-3}$ respectively, and letting $\tau \to t$, we find that $(\tau-t)^{-1}\,\sigma_i(\tau,t) \to p_i(t)$ uniformly, where

$$z^\alpha(t) \equiv p(t_\alpha),\quad z^\lambda(t) \equiv c_{f(\,)}^{\lambda\nu\beta\gamma}\,p_\nu(t)\,p_\beta(t)\,p_\gamma(t). \tag{8·13}$$

This completes the proof of (2) and (b). Now, $z^\lambda(t)$ is continuous by construction, and so are $z^\alpha(t)$, $c_{f(t)}^{\lambda\nu\beta\gamma}$. It follows that the $p_i(t)$ are continuous, proving (3).

For the proof of (c) it has to be shown that, for fixed t, $\partial\sigma_j(\tau,t)/\partial\tau$ tends to a limit as $\tau\to t$. For this, the following will suffice. We have

$$\partial\xi^i F(\tau,t)/\partial\tau = \sum_j (\partial\epsilon^i/\partial y_j)\,(y_1,\ldots,y_n,f(t))\,.\,\partial\sigma_j/\partial\tau,$$

where $y_i=\sigma_i(\tau,t)$. The $\partial\epsilon^i/\partial y_j$ can be expanded about $y=0$ by Taylor's formula, and their orders of magnitude obtained from (8·9). These, together with the fact that $\partial\xi^i F(\tau,t)/\partial\tau$, $(\tau-t)^{-2}\,\partial\xi^\lambda F(\tau,t)/\partial\tau$, $(\tau-t)^{-1}\,\sigma_i(\tau,t)$ all tend to finite limits as $\tau\to t$, lead to the required conclusion.

Finally, if $f=\pi\gamma$, where γ is a Θ-orbit, we have by corollary 2·3

$$f(\tau) = e_{f(t)}\tilde{f}_t(\tau) = e_{f(t)}\{(\tau-t)\,\gamma(t)\},$$

provided that $|\tau-t|<\epsilon$. Hence, $\tilde{f}_t(\tau)=(\tau-t)\,\gamma(t)$, so that $\dot{\tilde{f}}_t(t)$ is the isomorph of $\gamma(t)$. Accordingly, $q_f(t)=\gamma(t)$, which proves (4).

Theorem 8·1 enables one to approximate to a C''' H-arc f by a finite geodesic polygon which is homotopic to f by a well-known homotopy of Morse theory. The paths of this homotopy decrease in length as f is deformed into γ. We end with some simple applications of theorem 8·1.

Definition 8·2. A *co-path* in M is a piecewise C^0 map $p\colon R\to T^*$ such that πp is piecewise C^1 and $a\circ p(t)=(\pi p)_*\circ \mathrm{d}/\mathrm{d}t|_t$ for each $t\in R$.

In more classical language, a co-path is an H-path with a set of Lagrange multipliers, relative to some chart. From theorem 8·1 we can assert

Theorem 8·2. The parabolic structure $\mathscr{P}(M,H,a)$ determines a *canonical lift* (*a* co-path over f) $q_f\colon I\to T^*$ for any C''' arc f in M. In any chart this is equivalent to giving a set of Lagrange multipliers for f. With these multipliers, f satisfies the multiplier rule if f is geodesic.

Proposition 8·3. Let $f\colon I\to T^*$ be a co-path for which πf is C^1, and let $h\colon I\to T^*$ be a C^1 path for which $\pi h=\pi f$. There is an operation of *covariant differentiation* δ_f for f which takes h into a C^0 path $\delta_f[h]\colon I\to T$ such that $\pi'\delta_f[h]=\pi f$, $\pi'\colon T\to M$.

Proof. Let $U\{x^i\}$ be a chart in M containing im πf. For each $t\in I$,

$$f(t)=f_i(t)\,.\,\mathrm{d}x^i,\quad h(t)=h_i(t)\,.\,\mathrm{d}x^i$$

where f_i, h_i are C^0, C^1 functions of t, respectively. Define $\delta_f[h]$ for $U\{x^i\}$ by

$$\delta_f[h]\,(t)\circ\mathrm{d}x^i = \frac{\mathrm{d}}{\mathrm{d}t}[a(h(t),\mathrm{d}x^i)] + \Gamma^{jk,i} f_j(t)\,h_k(t),$$

$$2\Gamma^{jk,i} = a^{jk},_l a^{li} - a^{ij},_l a^{lk} - a^{ik},_l a^{lj},\; a^{ij}\equiv a(\mathrm{d}x^i,\mathrm{d}x^j),$$

where $\mathrm{d}a^{ij}=a^{ij},_k.\mathrm{d}x^k$. One checks in the usual way that $\delta_f[h]$ is independent of the choice of the chart $U\{x^i\}$. The operation δ_f extends to C^1 paths $h\colon R\to T^{(r)*}$, where $T^{(r)*}$ is a bundle over M of tensors of type $(0,r)$, turning them into paths in $T^{(r)}$ over πf. However, δ_f does not apparently extend to covariant or mixed tensors.

Definition 8·3. A C^1 path $h\colon I\to T^*$ over πf will be called *parallel on* f if $\delta_f[h]$ maps I into the zero section of T.

Definition 8·4. If $\sigma, \tau \in H_x$, call σ, τ *involutive* if $d\mu(\sigma,\tau) = 0$ for all $\mu \in \mathcal{N}_x$. For any $\tilde{\sigma} \in T^*_x$, set

$$Q_x(\tilde{\sigma}) = \{\tilde{\tau} \in T^*_x | a(\tilde{\sigma}),\, a(\tilde{\tau}) \text{ are involutive}\}.$$

Observe that $\dim Q_x(\sigma) = n-m$, $\dim a \circ Q_x = n-2m$ if $\operatorname{crk} H = m$.

Proposition 8·4. If $h: R \to T^*$ is parallel on f, $f(t)$, $h(t)$ are involutive for each t. Conversely, given a co-path f such that πf is C^2, and given $\sigma \in Q_x(f(0))$, there is a unique C^1 path $h: R \to T^*$ such that h is parallel on f and $h(0) = \sigma$, provided (as always) that $\operatorname{crk} H = m$.

Proof. The first statement is established by proving that $2\delta_f[h]\,(t) \circ \mu = d\mu(af(t), ah(t))$ for each $t \in R$ and $\mu \in \mathcal{N}$. The second statement is proved by a tedious piece of elementary analysis, which will be omitted.

Finally, we remark that, in the usual way, one can show that, if h_1, h_2 are parallel on f, then $a(h_1(t), h_2(t))$ is independent of t, i.e. parallel propagation preserves scalar products. Call a co-path which is parallel on itself an *autoparallel*. Geodesics are autoparallels, but not all autoparallels are geodesics. One can show that autoparallels in M are projections into M of orbits in T^* of vector fields Θ' satisfying

$$i[\Theta']\, d\omega + 2\, dL = \pi^*\mu,$$

where $\mu \in \mathcal{N}$ is arbitrary. One can also show that, if $\alpha: M \to T^*$ is a 1-form such that the orbits of $a(\alpha)$ are autoparallels, then these orbits are geodesics if and only if $d\alpha \equiv 0$.

Appendix 1

It is of interest to summarize the properties of the simplest case of a parabolic structure, namely $\mathscr{P}(H_o, N_o, \delta, a_o\}$ where (§ 4) H_o, a_o are the Euclidean plane and its metric, and $N_o = R$. For $\mu \in N_o$, $a_o(\delta\mu)$ is an infinitesimal rotation of H_o about the origin. With respect to a basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ in $R^3 = H_o \oplus N_o$, the tensors a, γ' have components

$$a^{ij} = \begin{pmatrix} 1 & 0 & -x_2 \\ 0 & 1 & x_1 \\ -x_2 & x_1 & x_1^2+x_2^2 \end{pmatrix}, \quad \gamma'^1_i = (x_2, -x_1, 1).$$

If (y_1, y_2, y_3) are co-ordinates in P^*_o relative to the dual basis $(\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3)$, the exponential map, obtained by integrating (2·4) using the above values for a^{ij}, is given by

$$\begin{aligned} 2y_3 e_o(y_i \mathbf{e}^i) = {} & \{y_1 \sin 2y_3 - y_2(1-\cos 2y_3)\}\, \mathbf{e}_1 \\ & + \{y_1(1-\cos 2y_3) + y_2 \sin 2y_3\}\, \mathbf{e}_2 \\ & + (y_1^2+y_2^2)\,\{1-(2y_3)^{-1} \sin 2y_3\}\, \mathbf{e}_3. \end{aligned} \qquad \text{(A 1)}$$

The set of horizontal paths in P which join the origin to a fixed point z have for their projections into H_o paths joining 0 to $\pi_h z$ and enclosing area $\tfrac{1}{2}\pi_n z$ with the straight line $0\pi_h z$. In particular, the projections of geodesics, having minimal length for given area, are circles.

From theorem 4·1, e_o restricted to the set $\{y_i \mathbf{e}^i \in P^*_o | y_1^2+y_2^2 > 0, |y_3| < 2\Pi\}$ is $1-1$; from (A 1) one checks that e_o is diffeomorphic on this set. One finds also from (A 1) that a *parabolic sphere*—the image under e_o of the set $\{y_i \mathbf{e}^i | y_1^2+y_2^2 = \text{const.}\}$ is a surface of revolution which cuts the $(\mathbf{e}_1, \mathbf{e}_3)$-plane as shown in figure 1. The shaded region exhibits the set covered by minimizing geodesics of fixed length issuing from O. This set is homeomorphic to a 3-ball (cf. corollary 2·4). The points of the parabolic sphere not bounding the shaded region are joined to the origin by non-minimizing geodesics. The latter project down onto circles in H_o covered more than once. The points represented by cusps and self-intersections are all images under e_o of conjugate points.

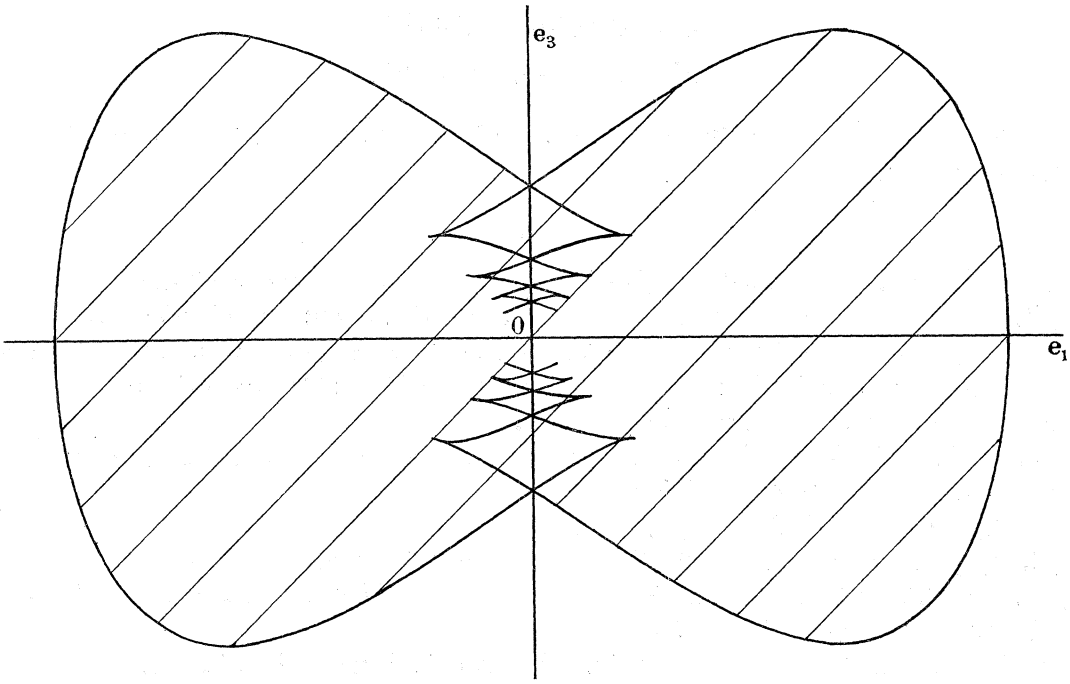

FIGURE 1

APPENDIX 2

For a Pfaffian system to be of maximal co-rank near $x \in M$, it is necessary and sufficient that, for constants $\lambda_{m+1}, \ldots, \lambda_n$ not all zero,

$$\Phi(\lambda) \equiv \mu^{m+1} \wedge \ldots \wedge \mu^n \wedge (\lambda_\nu \,\mathrm{d}\mu^\nu)^\sigma \neq 0, \quad \sigma = \tfrac{1}{2}m, \tag{A2}$$

where $\mu^{m+1}, \ldots, \mu^n$ is a basis for $\mathcal{N}_x$. In particular, m must be even. $\Phi(\lambda)$ is a homogeneous polynomial in the λ's of degree σ which, up to linear transformations, is an invariant of the Pfaffian system. For Φ to be non-zero for all non-zero λ, it must have even degree when $n-m>1$, so that $m \equiv 0 \pmod 4$. For any $\xi \in H_x$, $\xi \neq 0$, the vectors ξ, $a(\mathrm{i}[\xi]\,\mathrm{d}\mu^\nu)$, $\nu = m+1, \ldots, n$ are linearly independent, whence $n-m+1 \leqslant m$, $2m \geqslant n+1$. Not impossible values of m, n for $n \leqslant 10$ are as follows:

n	3	5	6	7	8	9	10
m	2	4	4	4, 6	4	4, 8	4, 8

Let S^{m-1} denote the unit sphere $\{\xi \in H_0 \mid |\xi| = 1\}$; then $(J_{m+1}, \ldots, J_n)$ where $J_\nu = a \circ \mathrm{d}\mu^\nu$, generates a field of $(n-m)$-frames on S^{m-1}. In particular if the J's are such that $J_\nu^2 = -1$, $J_\lambda J_\nu + J_\nu J_\lambda = 0$, they generate anti-commuting complex structures on H_o and the corresponding frames on S^{m-1} are orthonormal. By work of Eckmann (cf. Milnor 1963, §24) there are k anti-commuting complex structures on H_o, where $\dim H_o = r\theta_k$, $r = 1, 2, \ldots$, and

$$\theta_1 = 2, \quad \theta_2 = \theta_3 = 4, \quad \theta_4 = \theta_5 = \theta_6 = \theta_7 = 8, \quad \theta_8 = 16, \quad \theta_k = 16\theta_{k-8} \quad (k>8).$$

Each of these gives rise to a Pfaffian system of maximal co-rank on $H_o \oplus N_o$, where $\dim N_o = k$. For example, on $R^7 = R^4 \oplus R^3$ we have

$$\mu^5 = x^1\, dx^2 - x^3\, dx^4 + dx^5,$$
$$\mu^6 = x^1\, dx^3 + x^2\, dx^4 + dx^6,$$
$$\mu^7 = x^1\, dx^4 - x^2\, dx^3 + dx^7.$$

In this case, the endomorphisms 1, $a \circ d\mu^5$, $a \circ d\mu^6$, $a \circ d\mu^7$ of H_o ($\approx R^4$) are a basis for quaternion algebra.

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